One Parameter Model: Estimation and Inference for Proportions

Edps 590BAY

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ILLINOIS

Overview

- Pictures of how Bayes Theorem works
- Binomial distribution (likelihood, model for data)
- Beta distribution (prior and posterior)
- Explore combinations of beta and binomials
- Analytic results for estimation and inference of a proportion
- Posterior prediction
- Up-dating given new information
- Practice
- Grid method
- Comparing Two Proportions & a little Monte Carlo
- Practice

Depending on the book that you select for this course, read either Gelman et al. p29–30 or Kruschke Chapters 5 & 6.

Pictures of How Bayes Theorem Works

The posterior distribution of θ is a compromise between the prior and the posterior. Recall,

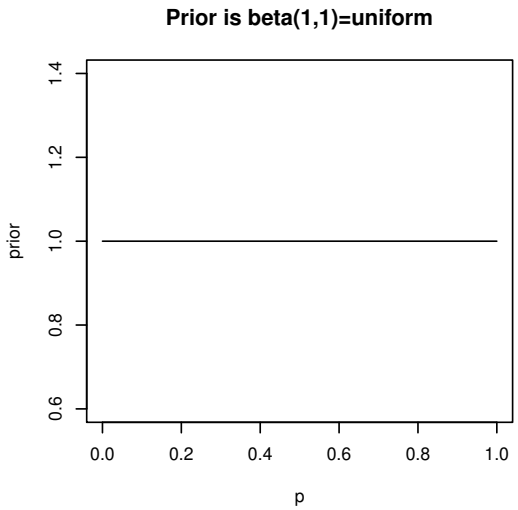
$$\begin{aligned} p(\theta|y) &= \frac{p(y|\theta)p(\theta)}{p(y)} \\ &\propto p(y|\theta)p(\theta) \end{aligned}$$

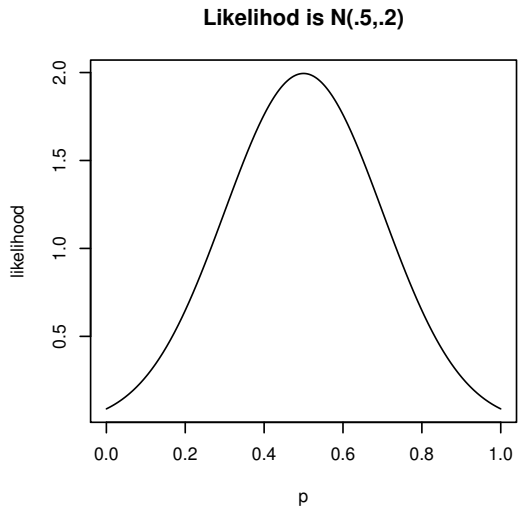
where $p(y)$

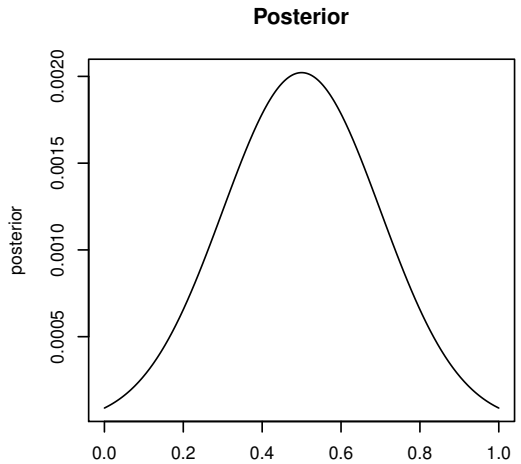
- ensures probability sums to 1 & is constant for given problem.
- "average" of numerator.
- For discrete y: $p(y) = \sum_{\theta \in \Theta} p(y|\theta)p(\theta)$
- For continuous y: $p(y) = \int_{\theta} p(y|\theta)p(\theta)d(\theta)$

Where y =number of "successes" out of n tries and θ = the unknown probability of success.

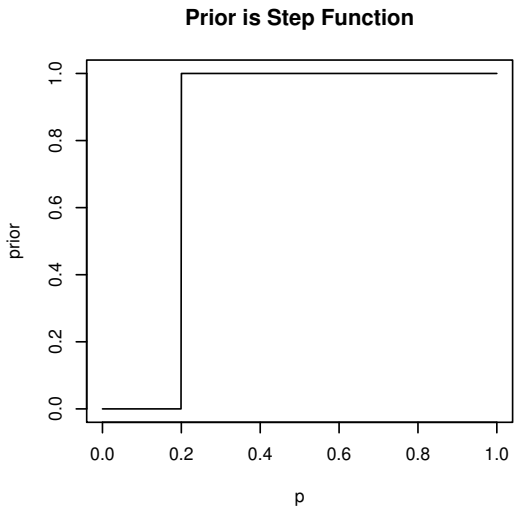
Example 1: No Prior Information

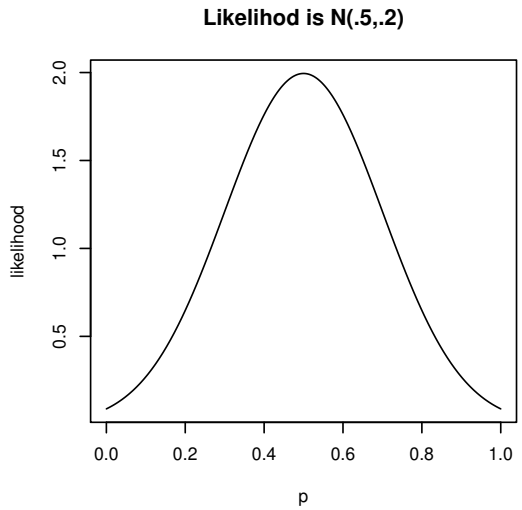


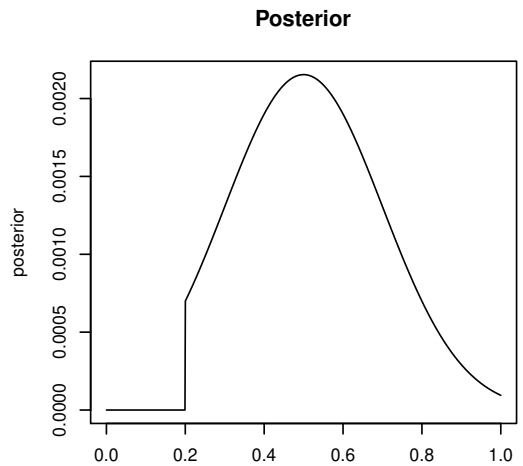




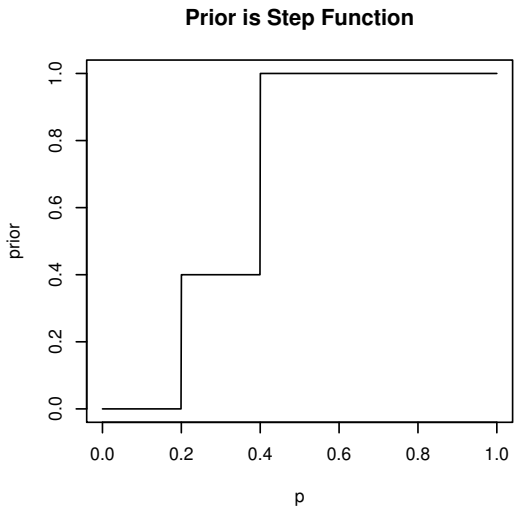
Example 2: Prior is Step Function

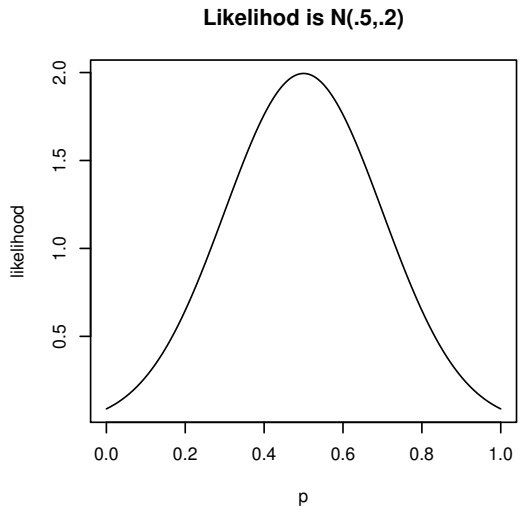


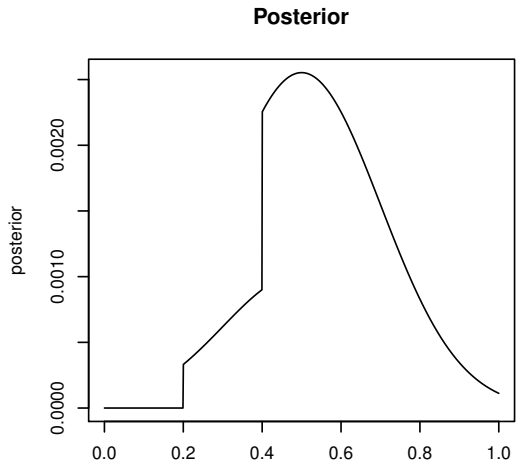




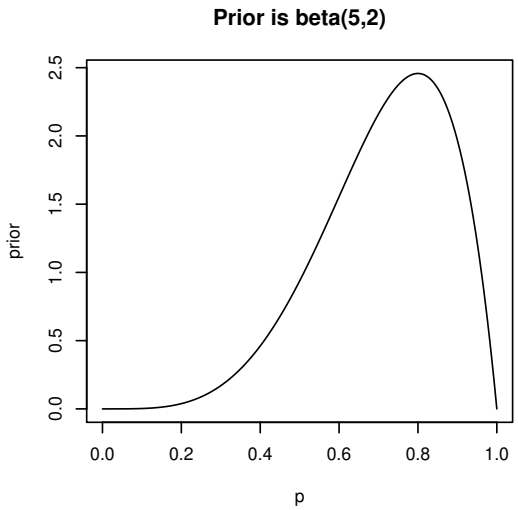
Example 3: Prior is a 2 Step Function

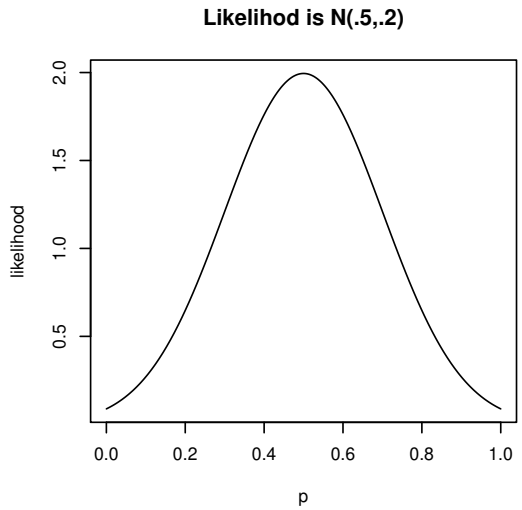


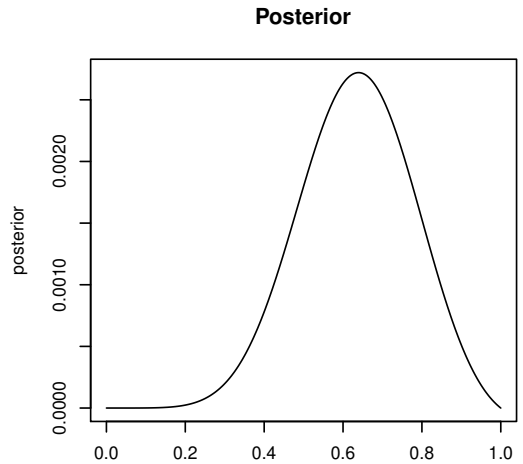




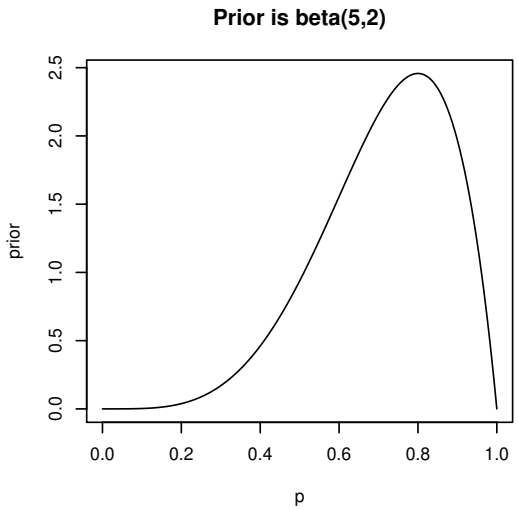
Example 4: Prior is Smooth Function

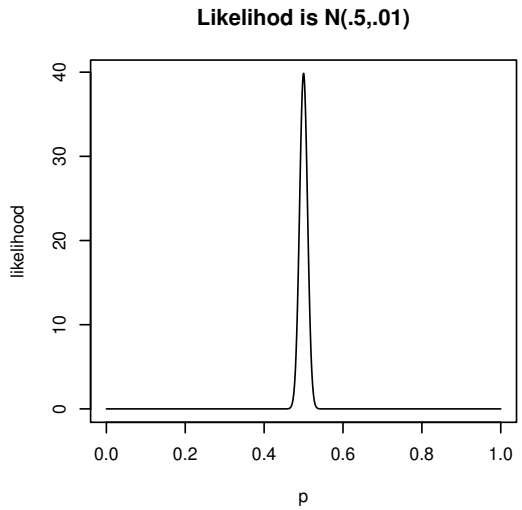


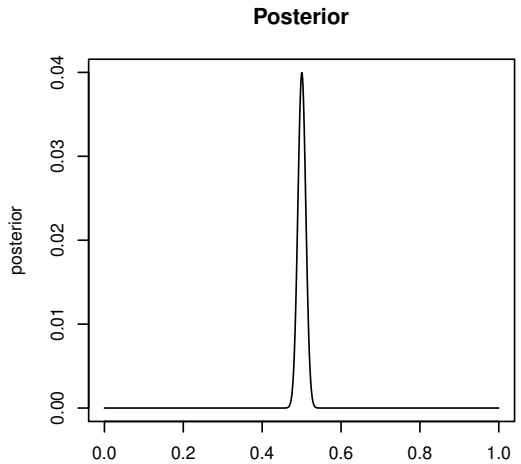




Example 5: Lots of data







Impact of prior and likelihood on posterior

Impact of prior and likelihood (data) on the posterior:

- Example 1: When you have no prior information ("ignorance"), the posterior distribution depends only on sampling model (i.e., the likelihood, the data)
- Example 2: With a step function, the posterior when prior value of the density equals 0, but once prior density equals 1, the posterior is the same as likelihood.
- Example 3: With a 2 step function, this is like the one step function, except for the 2nd step where the prior density is non-zero, the posterior between these corresponding ys is linear in shape with non-zero slope.
- Example 4: With a smooth prior and likelihood, the posterior is a "compromise" between them. For example, the mode of the posterior is between the mode between the prior and likelihood.
- Example 5: With lots of data, the posterior is mostly influenced by the likelihood.

The Binomial

For proportions, a natural probability mass is the Binomial. The Binomial will be our likelihood for the probability of a dichotomous variable; that is, our sample model.

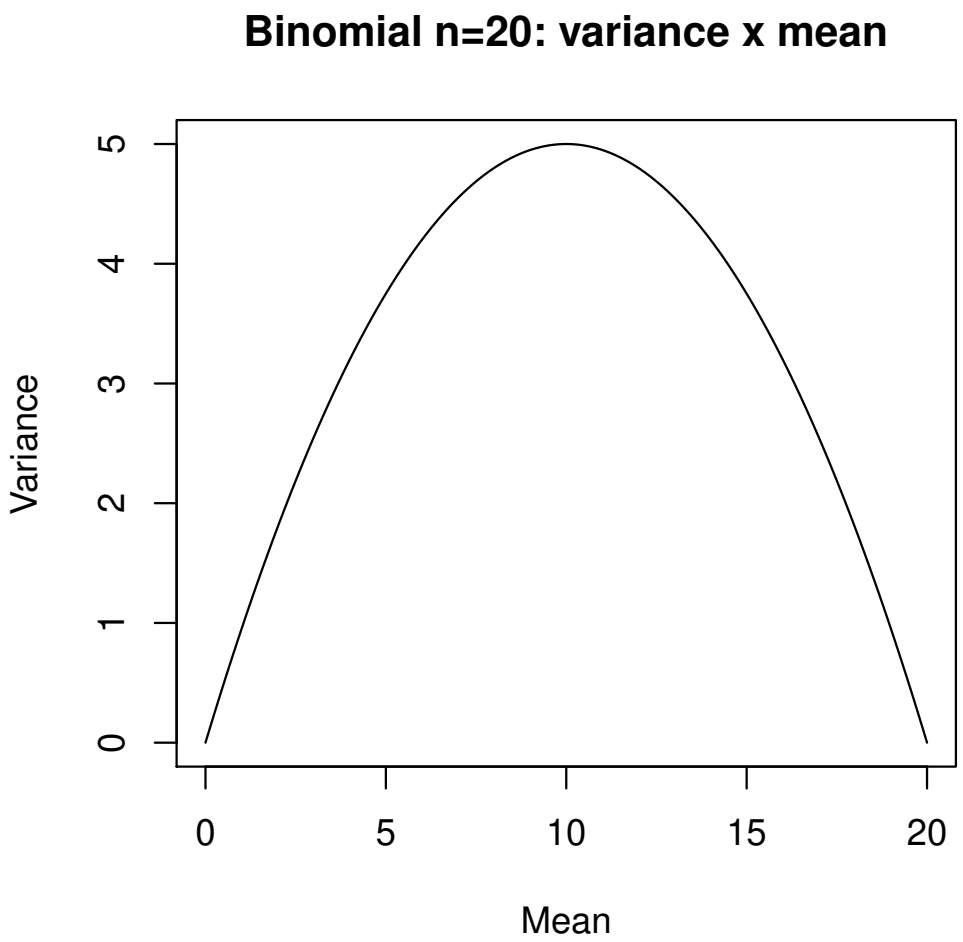
Let $y =$ number of "successes" out of n independent trials where the probability of a success is θ. The Binomial is

$$\begin{aligned} p(y|\theta) &= \binom{n}{y}\theta^y(1-\theta)^{n-y} \\ &= \frac{n!}{y!(n-y)!}\theta^y(1-\theta)^{n-y} \qquad \text{for } y = 0, 1, \ldots, n \end{aligned}$$

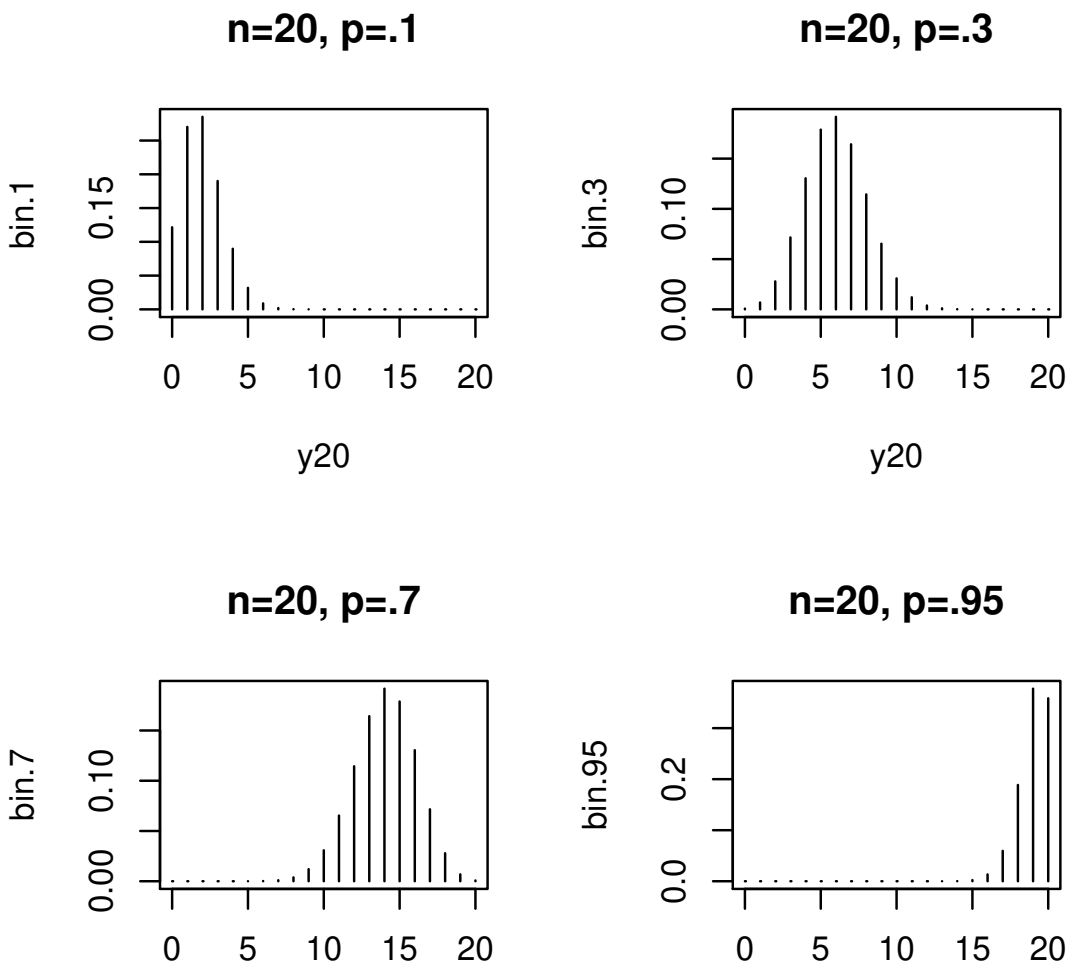
where

- Mean $\mu = n\theta$
- Standard deviation $\sigma = \sqrt{n\theta(1-\theta)}$

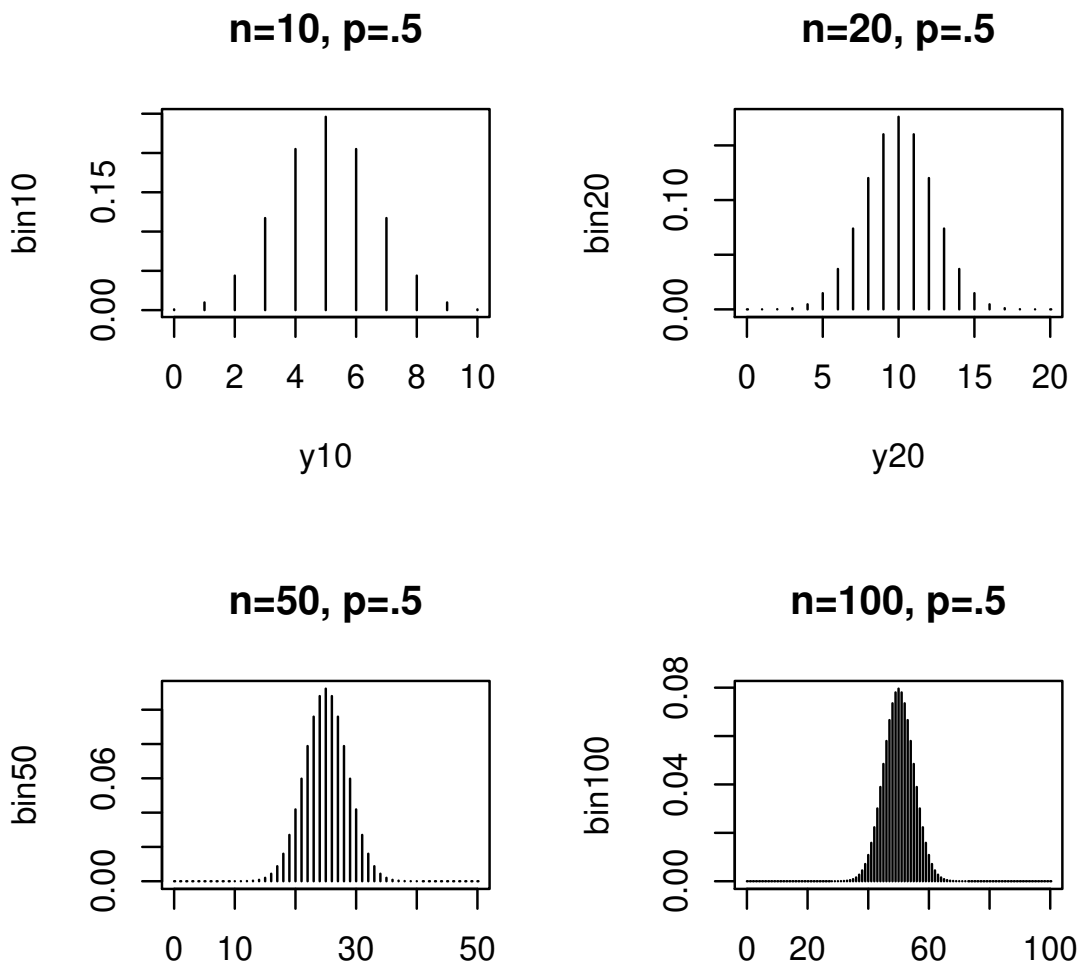
The Binomial: Variance as function of mean



The Binomial varying θ



The Binomial varying n



Prior: The Beta Distribution

Why?

- It has the same possible values as a probability:

$$0 \leq x \leq 1$$

- The density function can take on many different shapes.
- With a Beta prior and Binomial Likelihood, the posterior will be a Beta distribution.

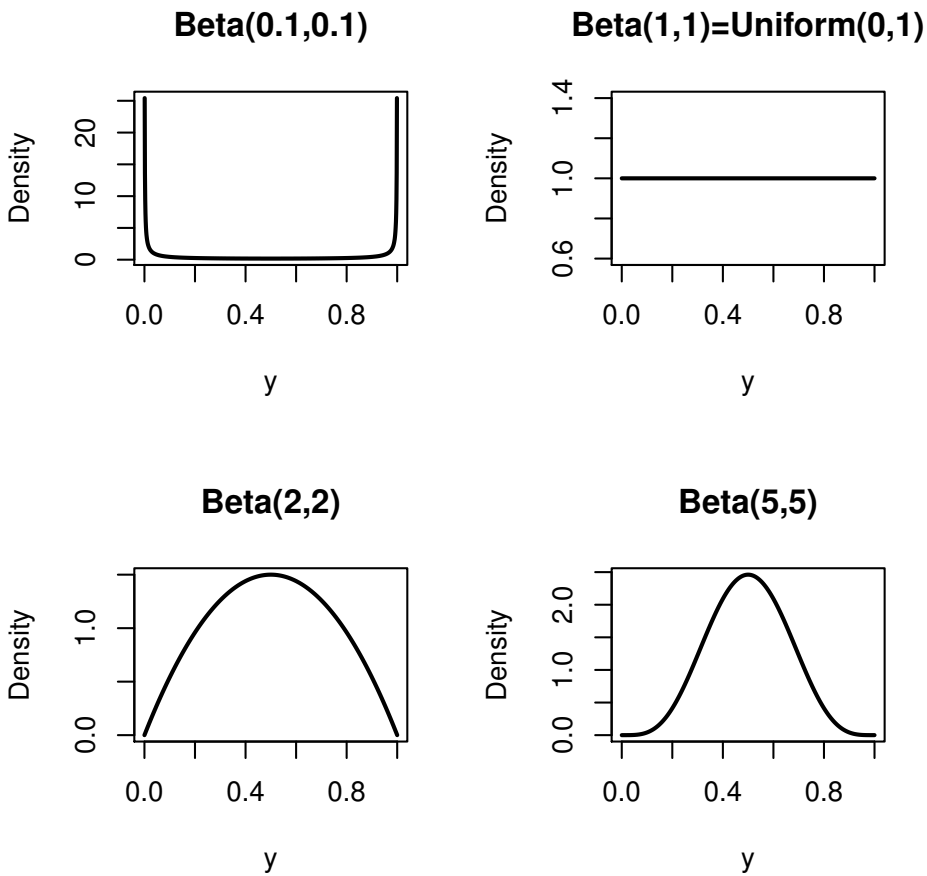
Definition: A Conjugate prior distribution is one where the product of the likelihood and prior (i.e., $p(y|\theta) \times p(\theta)$) yields a posterior distribution for θ that has the same form as the prior.

Conjugate Prior Distributions

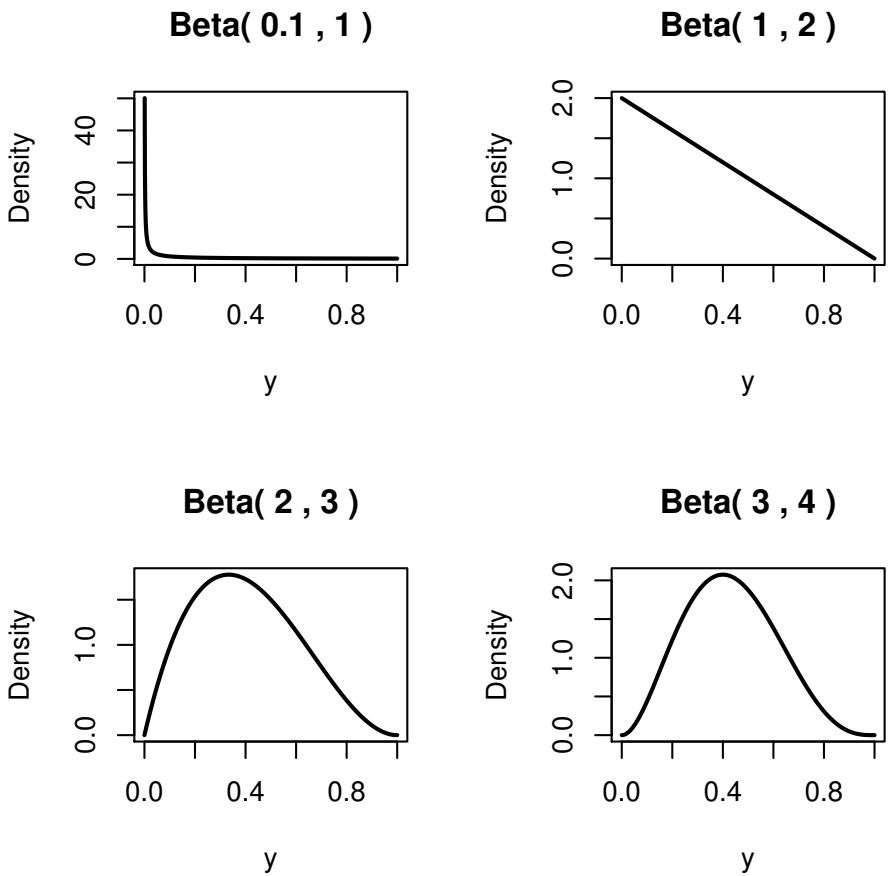
Data Distribution	Meaning of θ	Conjugate Prior for θ
Binomial	Proportion 'sucesses'	Beta
Multinomial	Proportion 'sucesses' (> 2 categories)	Dirichlet *
Normal	Mean	Normal or Uniform
Normal	Variance	Inverse-Gamma
Poisson	Rate (or count)	Gamma
Multivaiate Normal	Precision matrix	Wishart
Multivariate Normal	Covariance matrix	Inverse Wishart

* pronunciations of "Dirichlet"

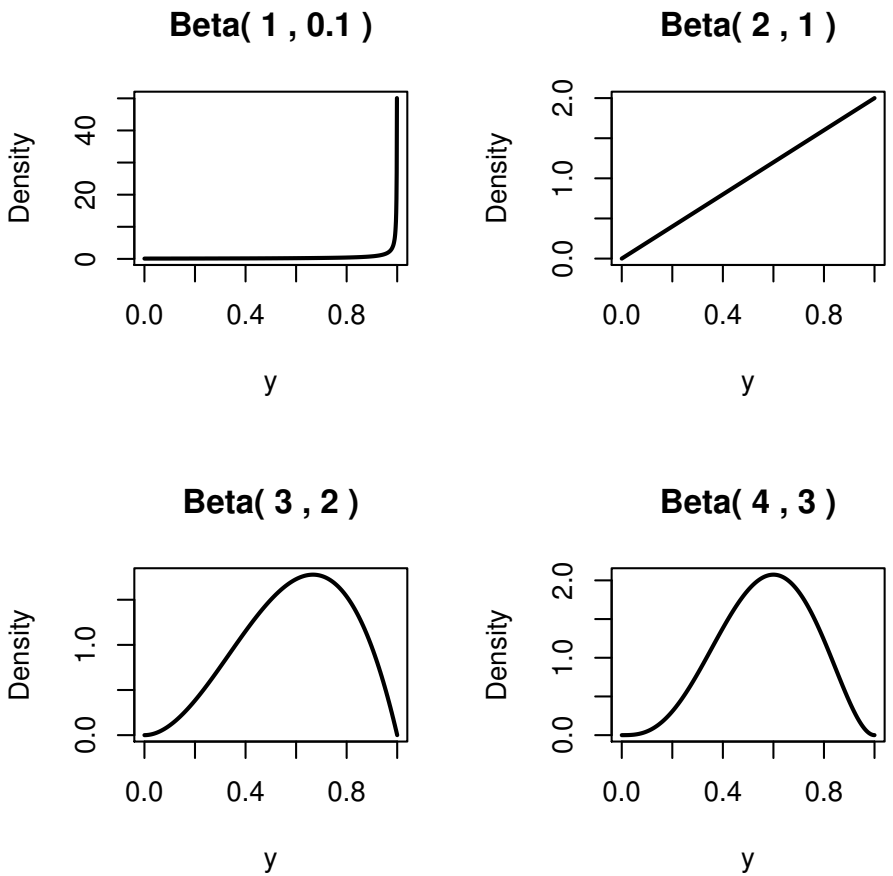
Examples of Beta Distribution: $a = b$



Beta with Right Skew: $a < b$



Beta with Left Skew: $a > b$



Facts about the Beta Distribution

- It depends on two parameters, a and b, where a is *like* the number of successes and b *like* the number of failures.
- Summary from figures:
  - $a = b \longrightarrow$ symmetric.
  - $a > b \longrightarrow$ left skew.
  - $b < a \longrightarrow$ right skew.
  - As a and b get larger, more centered around the middle.
  - What happens when $a < 1$ and $b < 1$ or $a < 1$ and $b > 1$? Run function from web-site.

$$\mu = \frac{a}{b+a}$$

$$\text{mode} = \omega = \frac{(a-1)}{(a+b-2)} \text{ for } a, b > 1$$

$$\text{“concentration”} = \kappa = a + b \quad \text{look at figues}$$

More notes on The Beta Distribution

Useful facts for Beta Distribution:

$$\begin{aligned}
\mu &= \frac{a}{b+a} \\
\text{mode} &= \omega = \frac{(a-1)}{(a+b-2)} \text{ for } a > 1 \text{ and } b > 1 \\
\text{median} &\approx \frac{a-1/3}{a+b-2/3} \text{ for } a > 1 \text{ and } b > 1 \\
\sigma^2 &= \frac{ab}{(a+b)^2(a+b+1)} \text{ for } a \text{ and } b > 1
\end{aligned}$$

The Beta Distribution (continued)

The Beta distribution is

$$\text{Beta}(\theta|a,b) = p(\theta|a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{(a-1)}(1-\theta)^{(b-1)}$$

where

- $\Gamma()$ is the "gamma function" — not "gamma distribution".
- $\Gamma(a) = \int_0^\infty t^{(a-1)}\exp(-t)dt$
- $\Gamma(a+b)/(\Gamma(a)\Gamma(b))$ is like the binomial coefficient for the binomial distribution.

Recall the Binomial distribution is

$$p(y|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y} = \left(\frac{n!}{y!(n-y)!}\right)\theta^y(1-\theta)^{n-y}$$

The Beta-Binomial

$$
\begin{aligned}
p(\theta|y,n) &\propto \text{(Binomial likelihood)} \times \text{(Beta distributon)} \\
&\propto p(y|\theta)p(\theta) \\
&\propto \binom{n}{y}\theta^{y}(1-\theta)^{n-y} \times \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{(a-1)}(1-\theta)^{(b-1)} \\
&\propto \frac{\Gamma(a+b+n)}{\Gamma(a+y)\Gamma(b+n-y)}\theta^{(y+a-1)}(1-\theta)^{n+b-y-1} \\
&\rightarrow \text{Beta}((a+y),(b+n-y)) \\
&= \text{Beta}(a^*,b^*)
\end{aligned}
$$

where $a^* = a + y$ and $b^* = b + n - y$.

Some Pictures of beta binomials

Play with R function that I wrote: beta.binomial

- Download from web-site (file name is "function_beta_binomial.txt")
- Copy and run in console
- Use: `beta.binomial(y,n,a,b)`
- Output: figure of prior ($p(\theta) = \text{Beta}(a, b)$), likelihood $p(y|\theta)$, and the posterior $p(\theta|y)$

OR

- Download and install package "LearnBayes"
- Use the command triplot(prior, data, where="_____")
  - prior = vector with scale parameters of beta
  - data = vector with y and $n - y$.
- For example

  triplot(c(3,2),c(7,3),where="topleft")

Mean of the posterior as a compromise

The mean of the posterior equals

$$
\begin{aligned}
E(\theta|y) &= \frac{y+a}{n+a+b} \\
&= \frac{y}{n+a+b} + \frac{a}{n+a+p} \\
&= \left(\frac{n}{n}\right)\frac{y}{n+a+b} + \left(\frac{a+b}{a+b}\right)\frac{a}{n+a+p} \\
&= \underbrace{\frac{y}{n}}_{\text{data}} \times \underbrace{\frac{n}{n+a+b}}_{\text{weight}} + \underbrace{\frac{a}{a+b}}_{\text{prior}} \times \underbrace{\frac{a+b}{n+a+b}}_{\text{weight}} \\
&= w\frac{y}{n} + (1-w)\frac{a}{a+b}
\end{aligned}
$$

"Shrinkage" is a property of Bayesian estimates. In the following
=.15in

Example: Heights of US Presidents

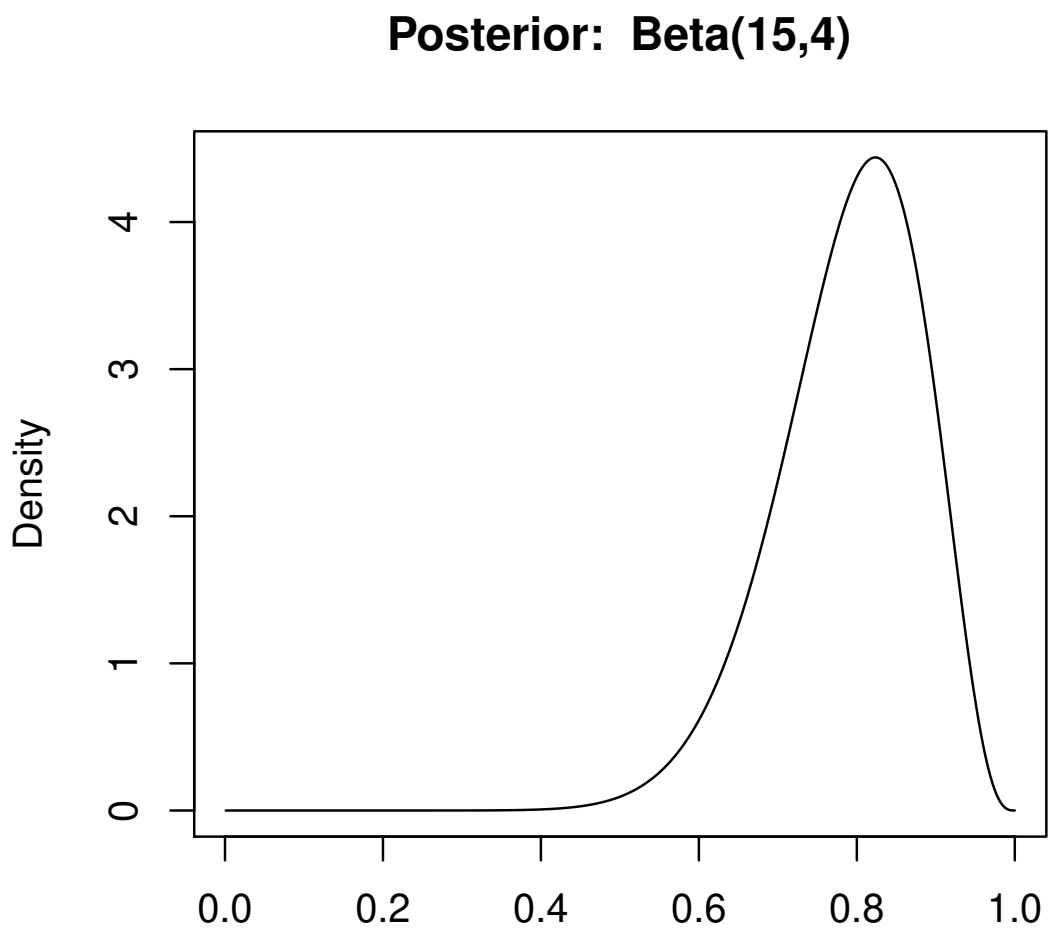
Data are from web-site en.wikipedia.org/wiki/Heights_of_presidents_and_presidential_candidates_of_ the_ United_States

- $y = 14$, the number times where the taller candidate won (if they were tied, I counted it as the taller winning) covering 1928-1996.
- $n = 17$, the number of elections (note: $14/17 = .8236$)
- We'll go with uninformative priors; that is, Beta$(1, 1)$ = Uniform$(0, 1)$
- The model could be stated as

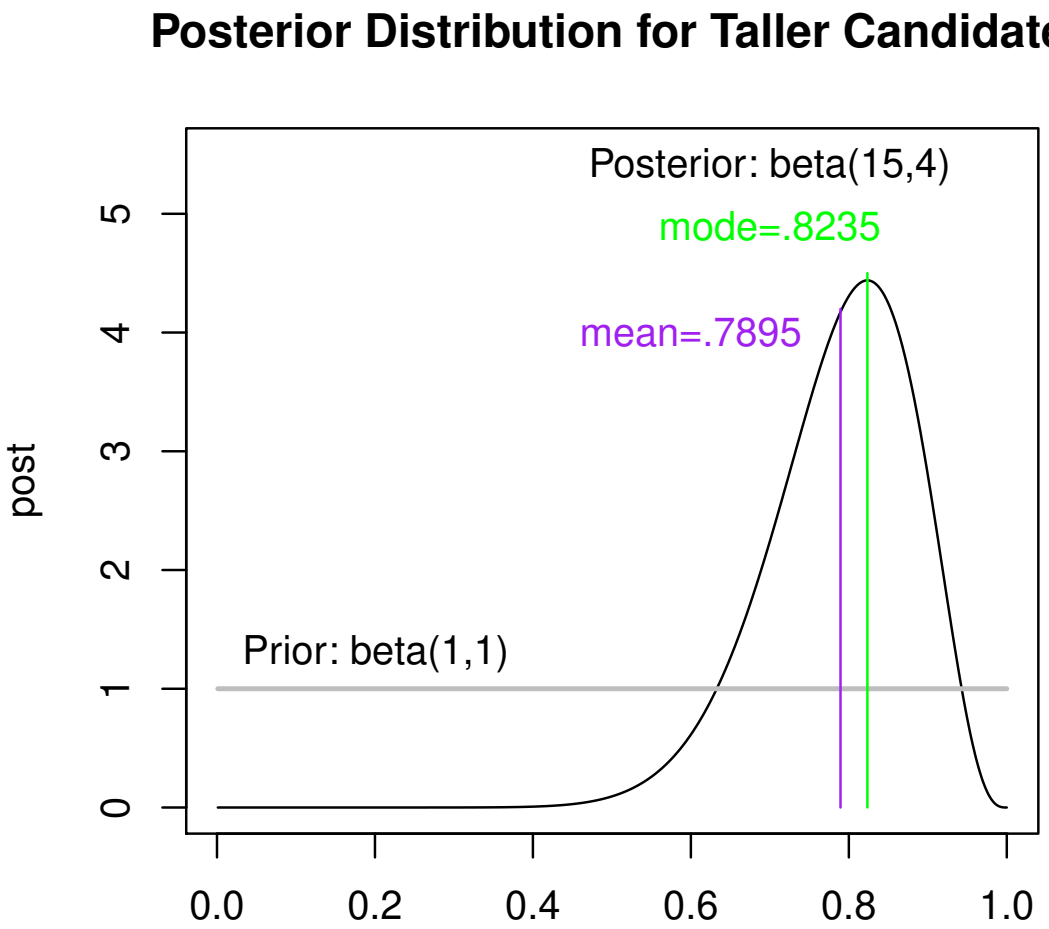
$$\begin{aligned} \text{likelihood} \quad & y \sim \text{Binomial}(\theta, n) \\ \text{prior} \quad & \theta \sim \text{Beta}(1, 1) \\ \text{posterior} \quad & \theta | y \sim \text{Beta}((14 + 1), (17 + 1 - 14)) = \text{Beta}(15, 4) \end{aligned}$$

which gives us mode$= ((15 - 1)/(15 + 4 - 2) = .8235$, mean$= 15/(15 + 4) = .7895$.

Posterior Distribution of Heights of US Presidents



Posterior Distribution of Heights of US Presidents



Credible Intervals

- Frequentist: Confidence interval ($L()$ and $U()$ are lower & upper limits)

$$Pr\left\{L(y) < \theta < U(y)|\theta\right\} = .95$$

  - Once we have data, the probability that the interval include θ is 0 or 1.
  - If we repeat process many times, then about 95% intervals constructed in the matter would include the true value of θ.

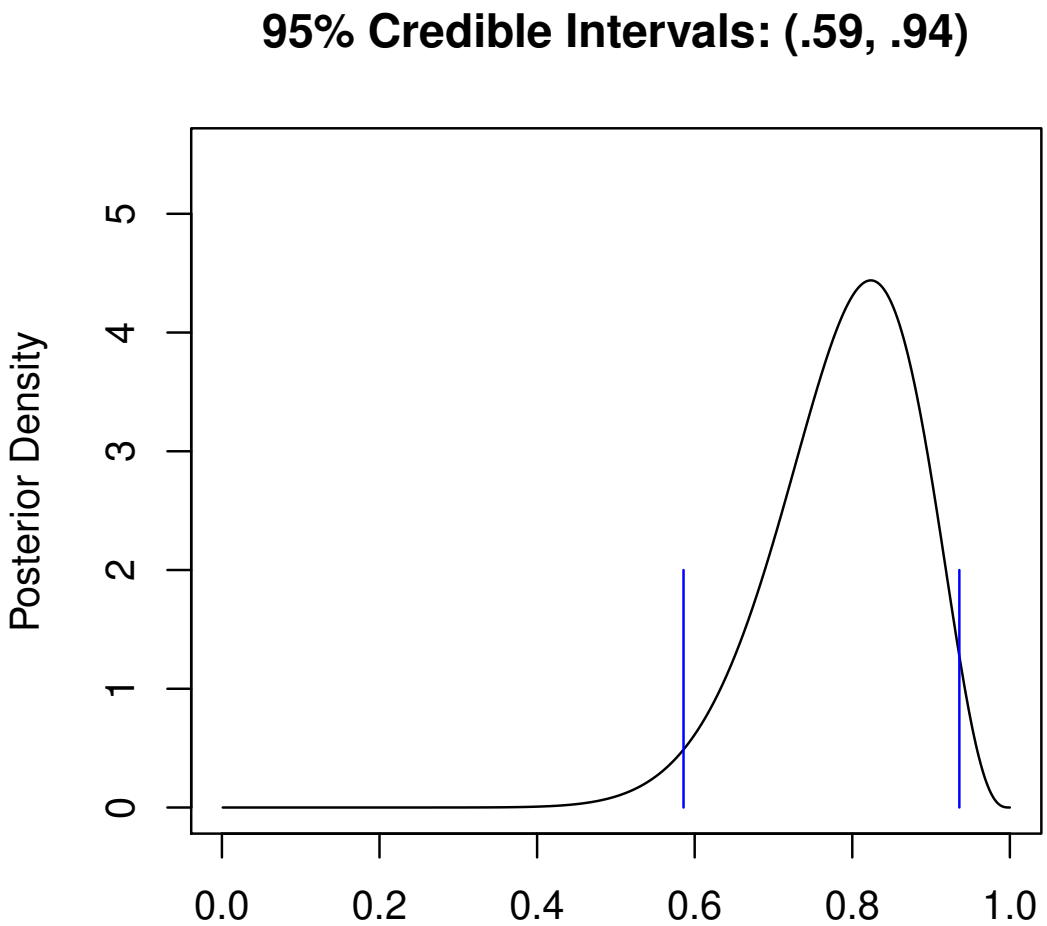
n

- Bayesian: Credible interval

$$Pr(L(y) < \theta < U(y)|y) = .95$$

  - There is a 95% chance that the interval from $L(y)$ to $U(y)$ includes the true value of θ.
  - This is conditioned on data (not unknown θ).

Credible Intervals



Computing Credible Intervals

```
# For 95% credible interval for taller
theta.95 ← c(.025,.975)
ci ← qbeta(theta.95,15,4)
ci
0.5858 .9359
```

High Density Intervals

- The credible intervals are wider.
- High density intervals are shortest ones with probability $(1 - \alpha)$.
- These have heights that are the same for both the left and right end point.
- They are harder to compute so we'll use package "HDInterval".

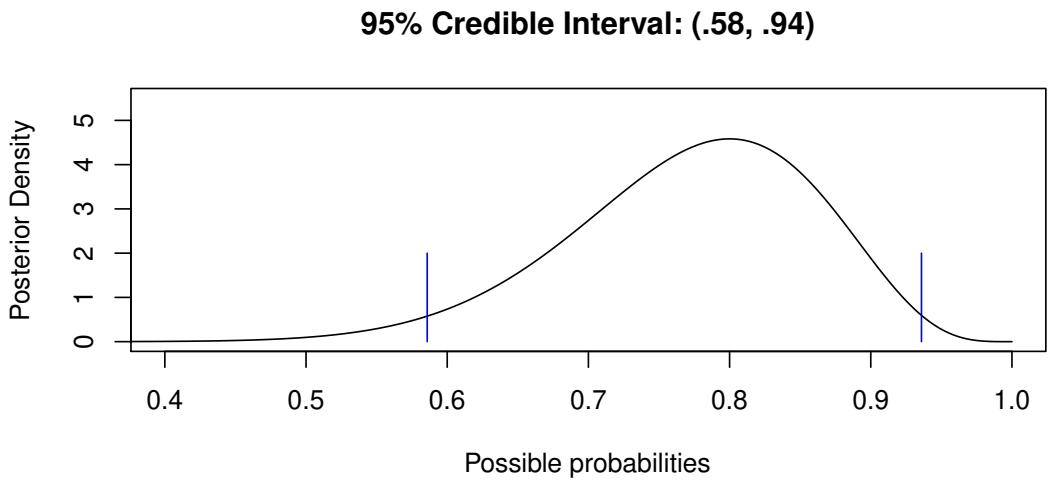
Finding High Density Intervals

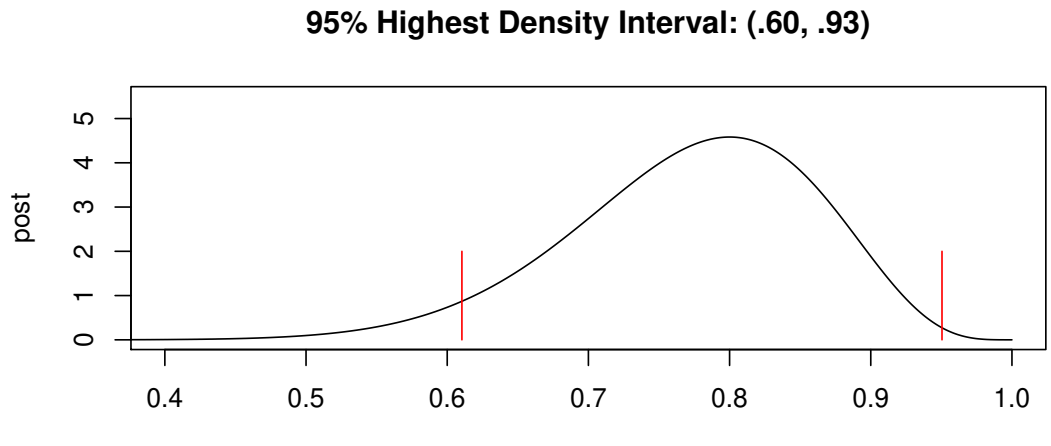
```
library(HDInterval)
inbeta ← rbeta(1E5,15,4)
hdi.p ← hdi(inbeta,credMass=.95)
hdi.p
     lower      upper
 0.5992848  0.9297222
attr(,"credMass")
```

Note lengths of respective intervals

- Credible Interval: $0.935908 - 0.5858225 = 0.3500854$
- High Density Interval: $0.9506483 - 0.6097860 = 0.3408623$

Compare HDI and Credible





Predicting New Observation

This include uncertainty about θ and inherent randomness in sampling.

$$
\begin{aligned}
p(\tilde{y}=1|y_1,\ldots,y_n) &= \int_0^1 Pr(\tilde{y}=1|\theta,y_1,\ldots,y_n)Pr(\theta|y_1,\ldots,y_n)d\theta \\
&= \int_0^1 Pr(\tilde{y}=1|\theta)Pr(\theta|y_1,\ldots,y_n)d\theta \\
&= \int_0^1 \theta Pr(\theta|y_1,\ldots,y_n)d\theta \\
&\equiv E(\theta|y_1,\ldots,y_n) \\
&= a^*/(a^*+b^*)
\end{aligned}
$$

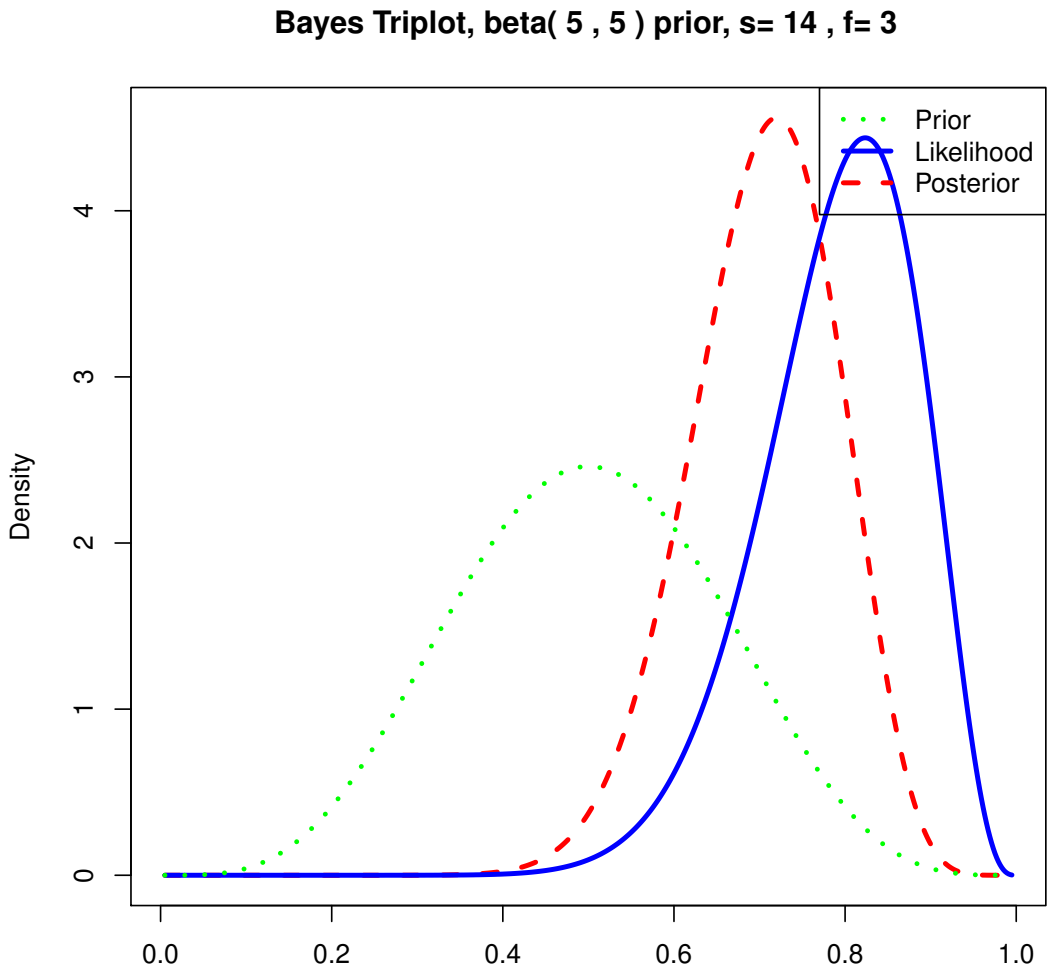
Where a^* and b^* are from the posterior distribution of θ given data. For our example where the posterior is Beta$(15,4)$ so
$Pr(\tilde{y}) = 15/(15+4) = 15/19 = .7895$ — the mean of the posterior.

Using a prior Knowledge

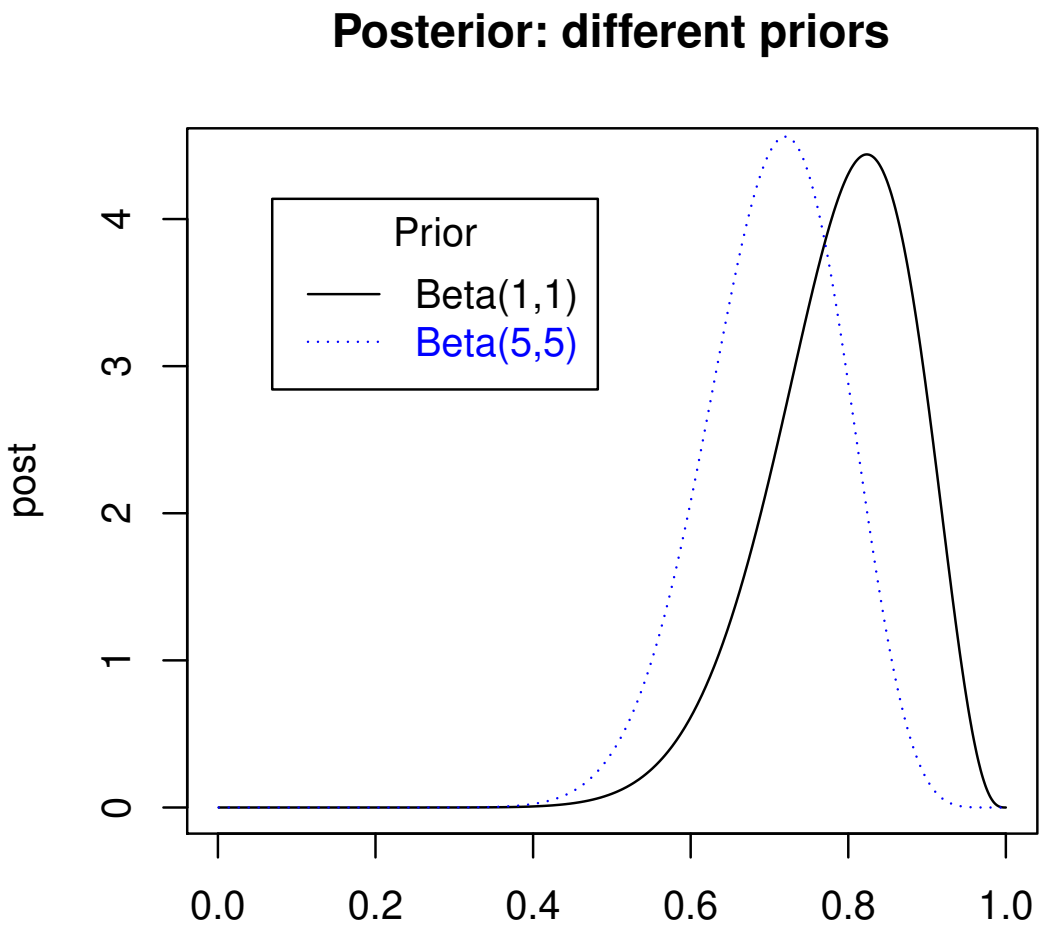
Suppose I think that the probability of taller wining is $1/2$, and I have a fairly strong belief I could choose a beta

- $\theta = .5$.
- Choose larger values for a and b
- For our example, if we could use Beta(5,5).
- The posterior distribution would be Beta($y + 5$,$n - y + 5$).
- For our example,Beta$((14 + 5), (17 - 14 + 5))$ = Beta$(20, 7)$
- Mean of the posterior would be $20/(20 + 7) = 20/27 = .7407$.
- Mode of the posterior would be mode$=(20 - 1)/(27 - 2) = .7600$.

Using Informative Prior



Different Posteriors using Informative Prior

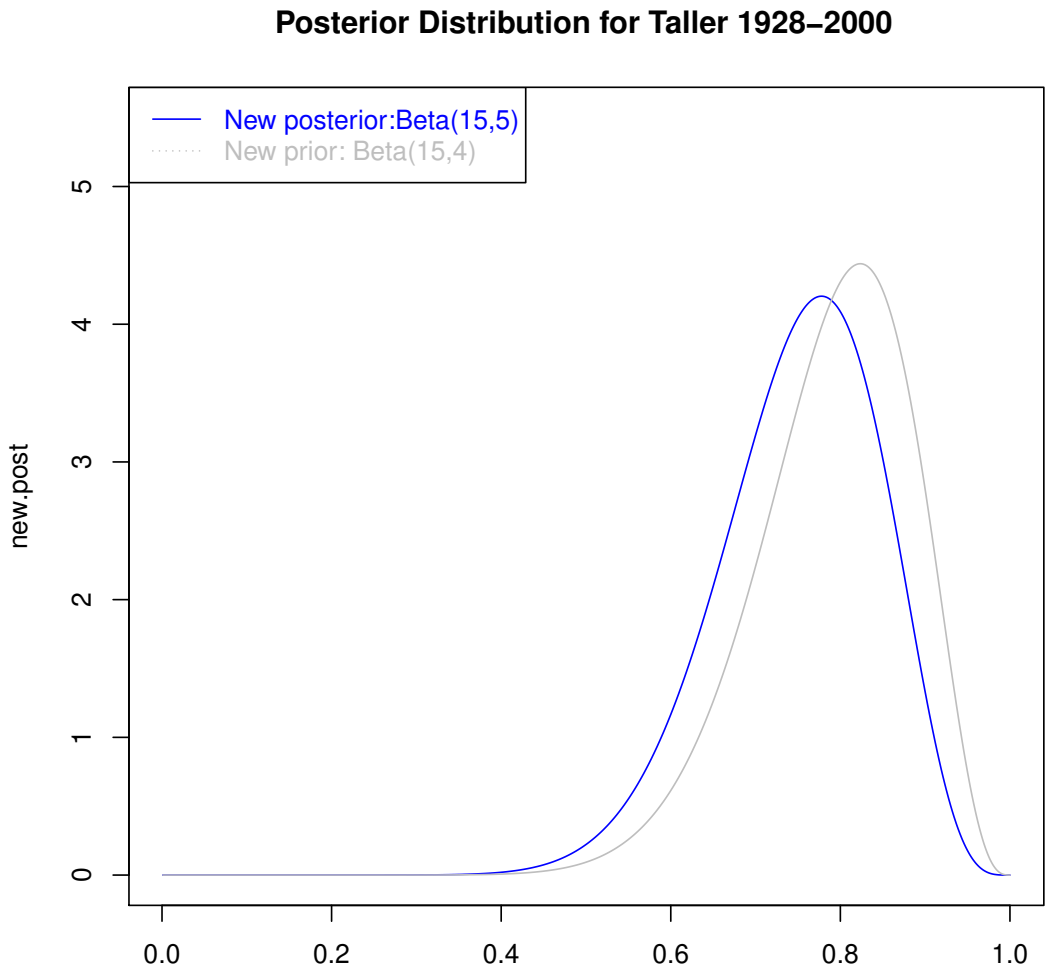


Up-dating and Using New Information

I only used elections up through 1996. How would our posterior change?

- Our posterior based on data is used as our prior to get new posterior.
- In 2000, the taller candidate lost: Bush vs Gore and Gore was taller.
- Current posterior is Beta$(15, 4)$, our new posterior is Beta$(15 + 0, 4 + 1) =$ Beta$(15, 5)$
- New mean $(15/20) = .75$
  new mode $= (15 - 1)/(15 + 5 - 2) = .78$
  new median $\approx (15 - 1/3)/(20 - 2/3) = .76$.
- Data values are Exchangeable. The subscripts are just considered labels with no meaning (i.e., order doesn't matter).

Adding Another Observation



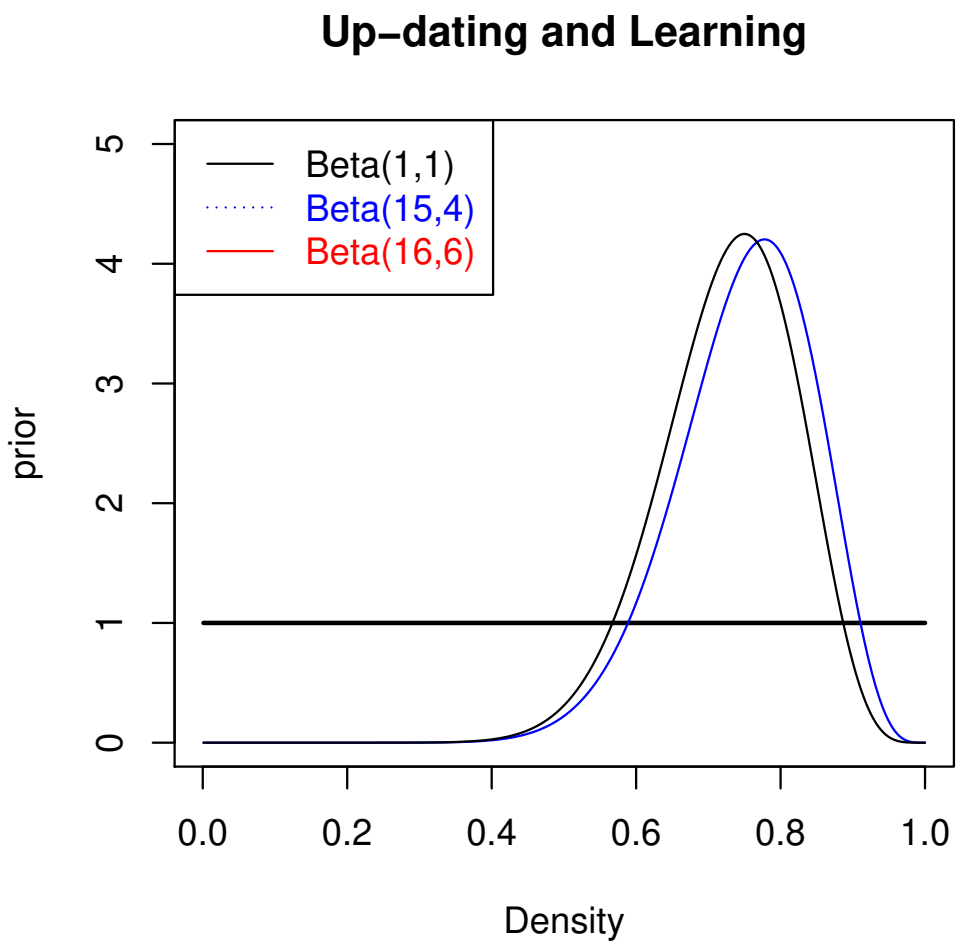
Using Even More Information

Lets add elections 2000, 2004, 2008 and 2012 to our original data set:

Year	Winner	cm	Loser	cm
2012	Obama	185	Romney	187
2008	Obama	185	McCain	175
2004	GWBush	182	Kerry	193
2000	GWBush	182	Gore	185

So started with $y = 14$ and $n = 17$ and a uniform prior, which gave us a posterior Beta$(15, 4)$. We use this as our prior. New data has 1 case where taller won and 3 cases where the shorter won.
Our new posterior is Beta$(15 + 1, 4 + 3)$ with a mean of $16/(16 + 7) = .6957$ and a mode of $(16 - 1)/(16 + 7 - 2) = .7143$,

Learning with More Data



Model Checking

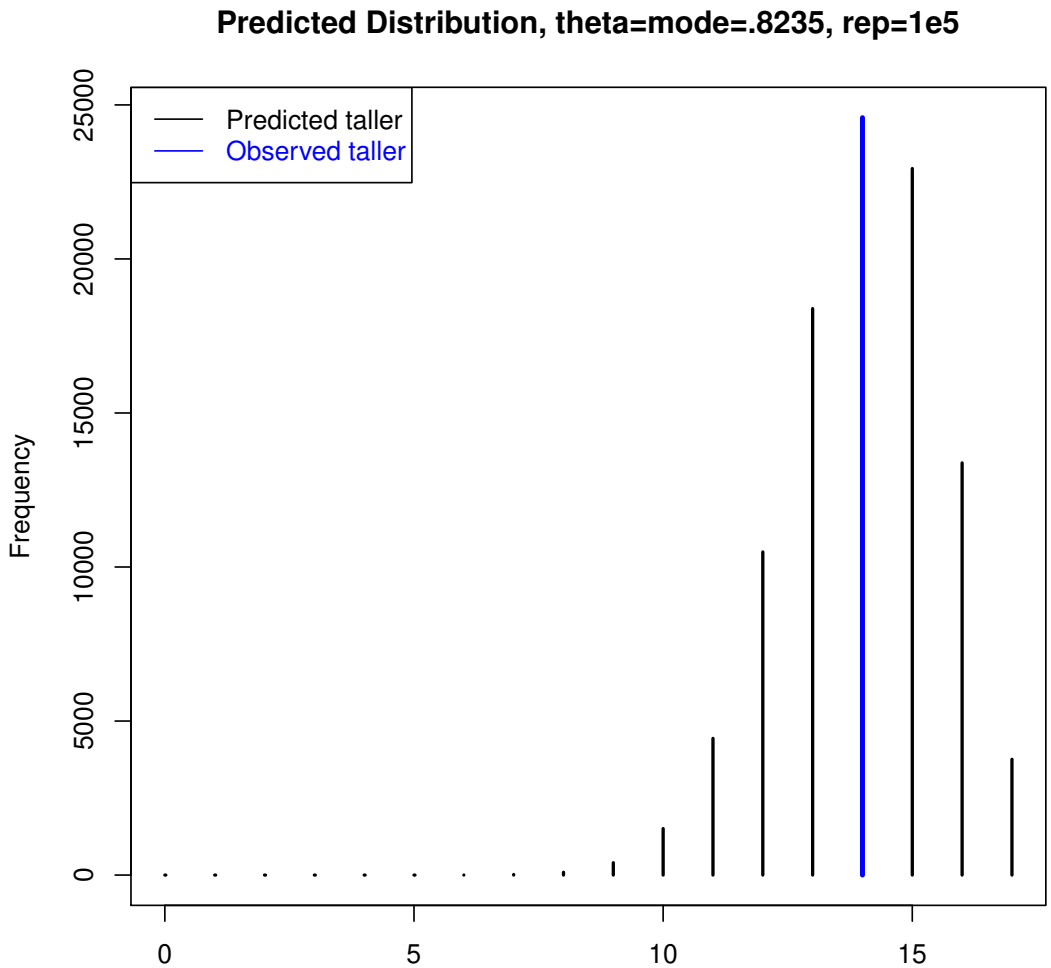
We'll check our original model: $y = 14$, $n - 7 = 3$, $n = 17$, uniform priors with Beta$(15, 4)$ are posterior distribution for θ.

We will draw $100,000$ samples of size 17 from our posterior distribution Beta$(15, 4)$ where $\theta = .8235$, the mode.

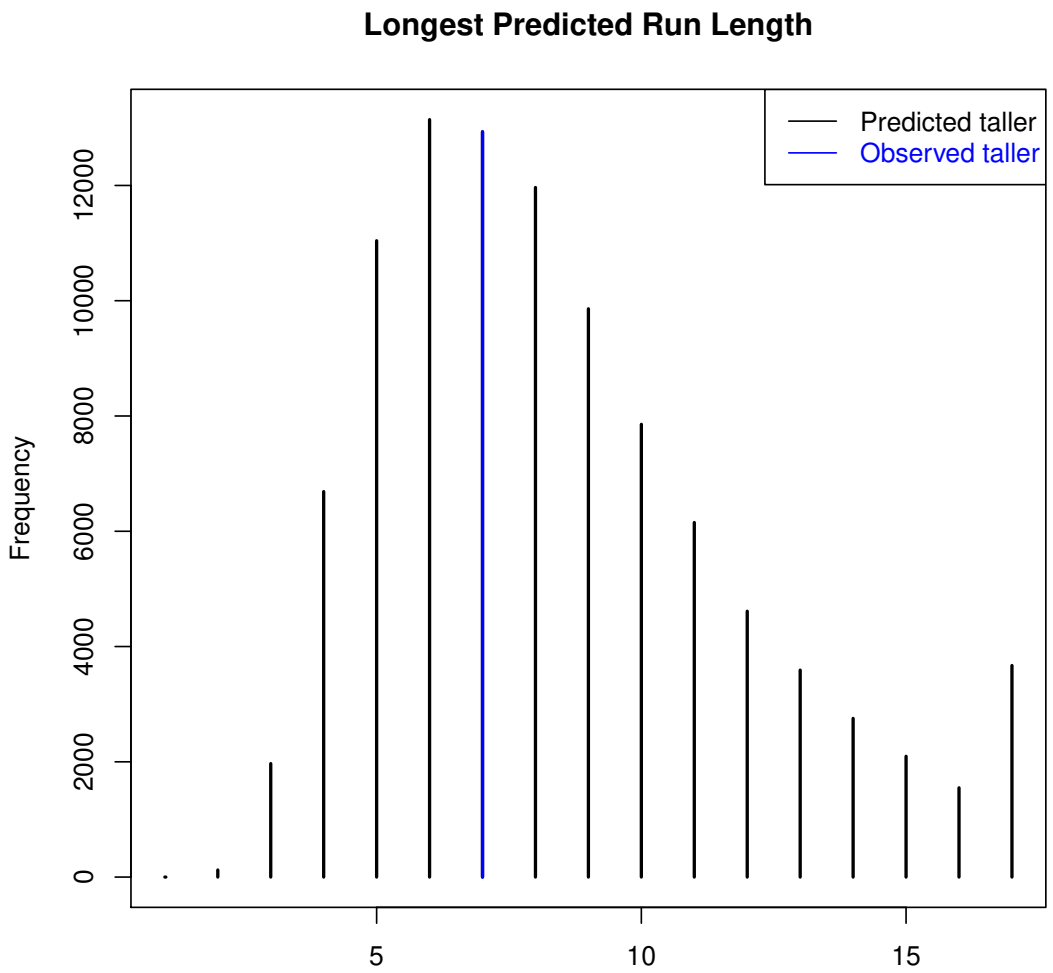
- How well do random samples from the posterior match our data?
- Does the predicted maximum run length match our data?
- Does the number of switches for predicted match the number in the data?

The latter two are crude ways to examine autocorrelation.

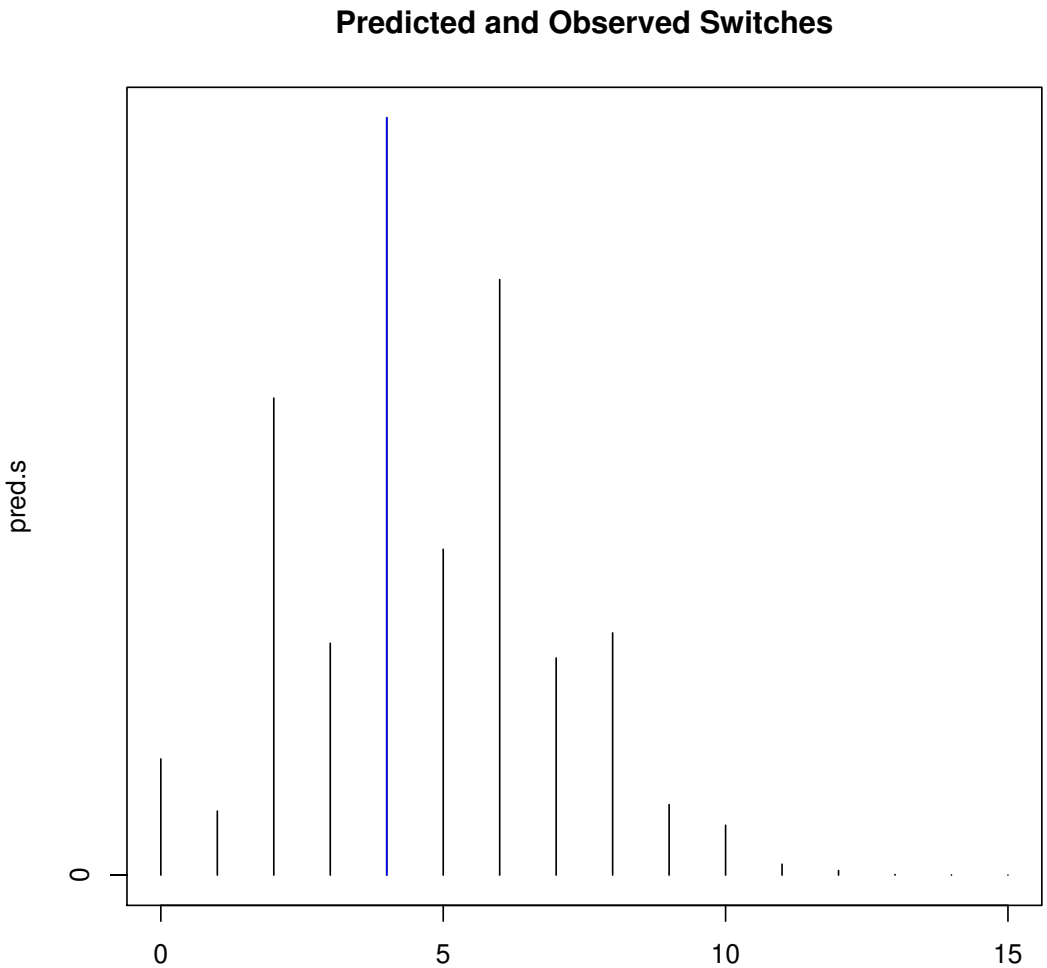
Data and Random Sample from Posterior



Max Run Length



Number of Switched (need to check)



Practice: Covid19

You will need to download 2 data sets for this:

cases.csv

deaths.csv

And the Rmarkdown document. If you want to save your work, you might also want to use the Rmd file and edit it.

Practice 2

Just about everything we did in this lecture; however, using data from book *Applied Bayesian Statistics* by M.K. Cowles.

"On March 15, 2002, the *Iowa City Press Citizen* carried an article about the intended 19% tuition increase to go into effects at the University of Iowa (UI) for the next academic year. Let's revisit that time and suppose that you wish to send the regents and the stat legislature some arguments again this idea. To support your argument, you would like to tell the regents and legislators what proportion of current UI students are likely to quit school if tuition is raised that much. . . . pick a simple random sample of $n = 50$ students from the student directory are ask each of them whether she or he would be likely to quit school if tuition were raised by 19%." 7 students said that would have to quit school.

Practice

Using a uniform prior,

- What is the posterior distribution? (R not required)
- What is the mean of the posterior distribution? (R not required)
- What is the median of the posterior? (R not required)
- Plot the posterior. Are the mean and mode where you expect they would be? (R: seq, dbeta, plot or use either beta binomial function or triplot)

Practice

- Using a uniform prior (continued),
  - Plot the credible intervals on same graph as posterior. (R: seq, dbeta, qbeta,)
  - Plot the high density intervals. (R: seq, hdi, dbeta, plot)
- What is the probability that a new sampled student would have to quite school (R: not required)
- What is your new posterior distribution? (R: not required)
- Draw random samples from the new posterior and compare with your observed data. (R: seq, rbinom, plot, lines)

Normal Approximation

- Since n is large, the distribution of y, which is a sum of n observations, becomes approximately normal.
- For y binomial, can use
  - $\hat{p} = y/n$ as an estimate of the mean
  - $\sqrt{n\hat{p}(1-\hat{p})}$ as an estimate of the standard deviation.
- Also compute exact distribution (i.e., use `binom.test(y,n,conf.level=.85))`
- Compare the estimates of θ from Bayesian and frequenist methods.

Grid Method of Estimation of $p(\theta|y)$

1. Define grid (i.e., points to use to estimate θ, which is a probability)

```
grid <- seq(0,1,length.out=50)
```

2. Compute value of prior at each point on grid; e.g., for beta prior (need to put in numbers for a and b)

```
prior ← dbeta(grid,a,b)
```

3. Compute value of likelihood at each point on grid; e.g., for $n = 20$ and $y = 14$

```
likelihood ← dbinom(y,n,prob=grid)
```

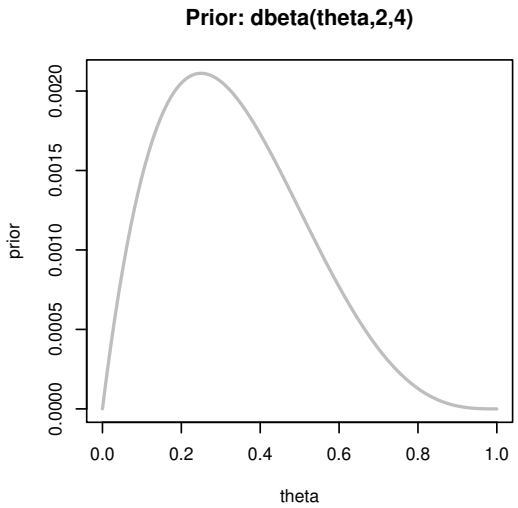
4. Compute unstandardized posterior

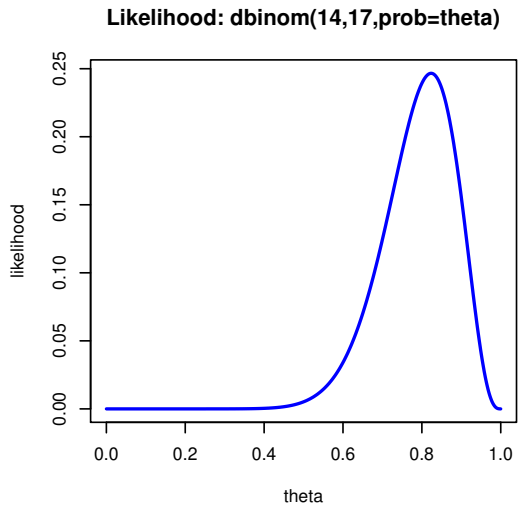
```
posterior.unsd ← prior*likelihood
```

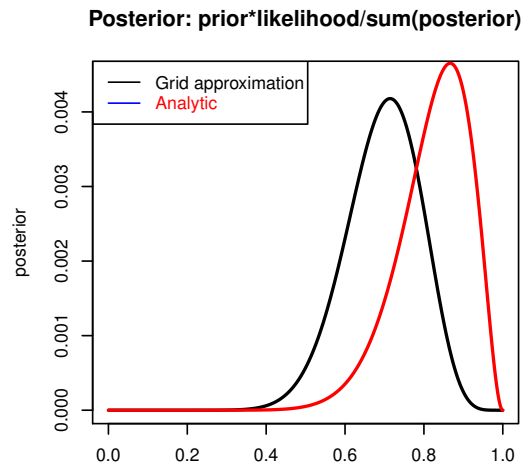
5. Standardize posterior

```
posterior ← posterior.unsd/sum(posterior.unsd)
```

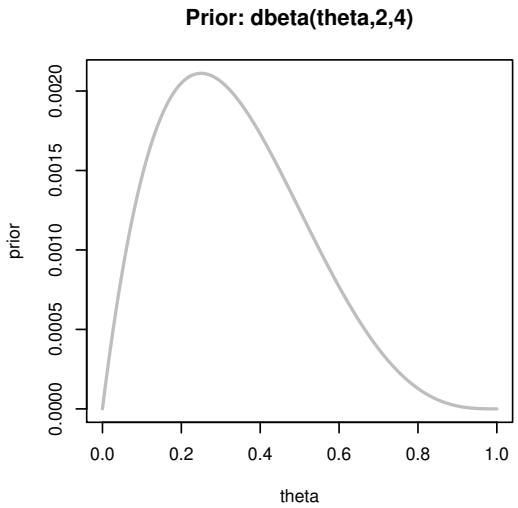
Example: Grid approximation

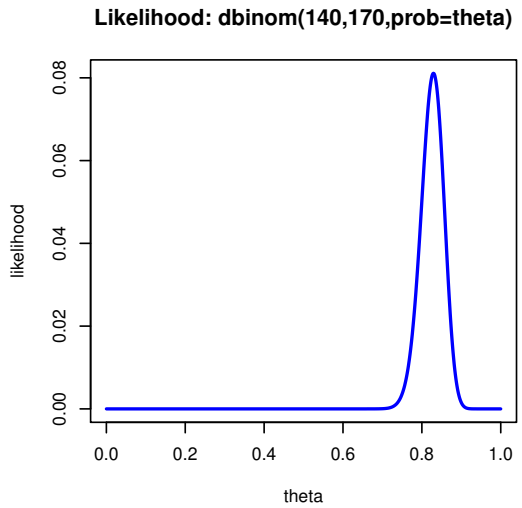


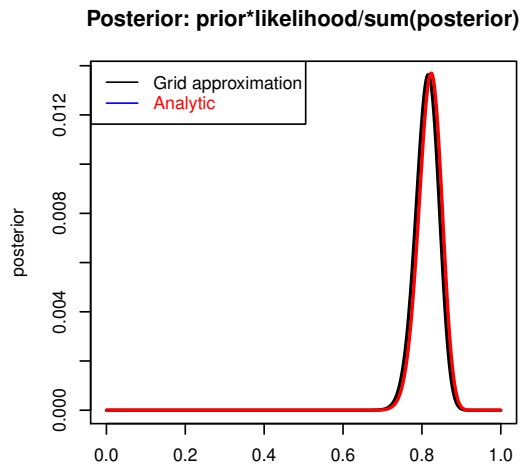




Larger Sample







Using grid.BetaBin

Download the function "grid.BetaBin" from the web-site and run it to examine the impact of

- sample size
- prior (i.e., a and b of the Beta distribution)
- y

Comparing 2 proportions

We can compute functions of θ and obtain their posterior distributions via Monte Carlo.
Possible functions of interest for a proportion could be

- Ratio: $\theta/(1-\theta)$
- Difference between two: $\theta_1 - \theta_2$
- Whether $\theta_1 > \theta_2$

We will focus on θ_i from two different populations.

We can accomplish this by using Monte Carlo simulation (sampling) from the posterior distributions for each population.

Monte Carlo

from wikipedia:

Monte Carlo methods (or Monte Carlo experiments) are a broad class of computational algorithms that rely on repeated random sampling to obtain numerical results.. . . In principle, Monte Carlo methods can be used to solve any problem having a probabilistic interpretation.

A Monte Carlo algorithm is an algorithm for computers which is used to simulate the behavior of other systems. It is not an exact method, but a heuristical one, typically using randomness and statistics to get a result. For us, we will sample from the posterior distribution (we did this previously for model checking).

Monte Carlo and 2 Proportions

- Let θ_1 be the unknown probability from population 1 (e.g., taller candidate winning election 1928–2016).
- Let θ_2 be the unknown probability from population 2 (e.g., taller candidate winning election 1789–1924).
- Find the posterior distributions for each population.
- Take S samples from the posteriors:

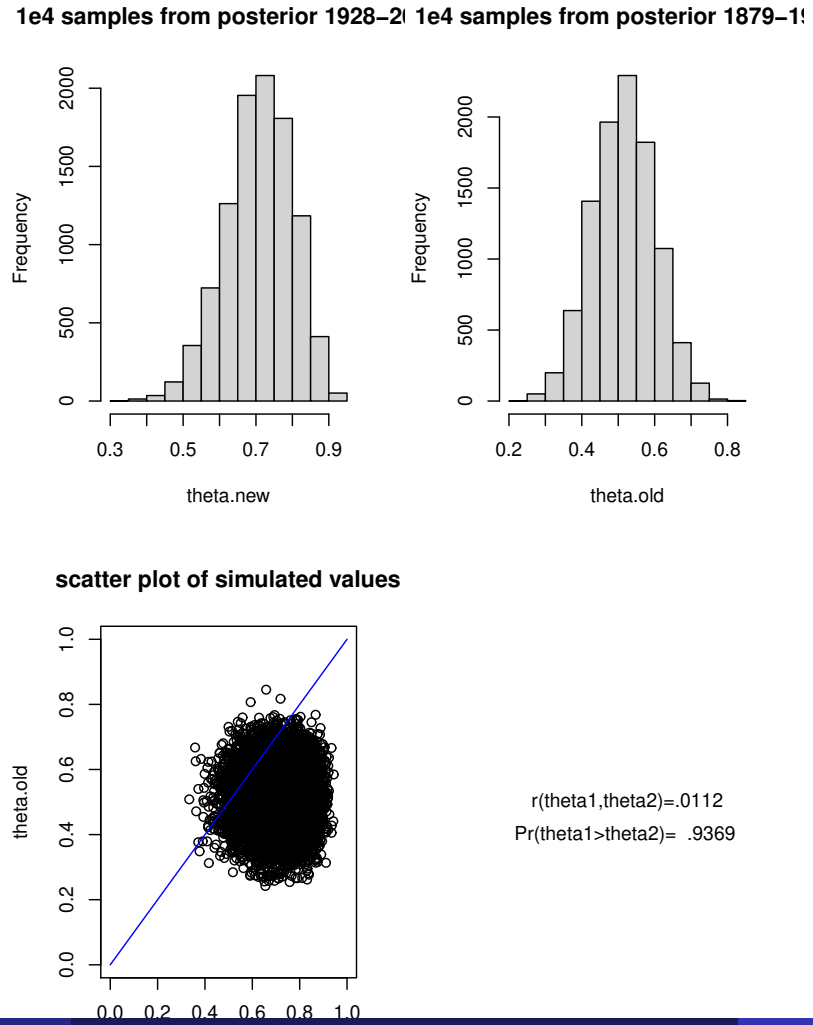
$$\begin{array}{cc} p(\theta_1^{(1)}|y) & p(\theta_2^{(1)}|y) \\ p(\theta_1^{(2)}|y) & p(\theta_2^{(2)}|y) \\ \vdots & \vdots \\ p(\theta_1^{(S)}|y) & p(\theta_2^{(S)}|y) \end{array}$$

  We now have S independent samples of joint posterior distribution of (θ_1, θ_2)
- Compute (for example) $Pr(\theta_1 > \theta_2)$ by counting how many times this happened out of the S samples.

Example of Two

- Question: Are US presidents more likely to be taller than the other candidate in more recent elections?
- For 1928–2016: $y_1 = 16$, $n_1 = 22$, so posterior is Beta$(17, 6)$ and mode $\theta_1 = .7273$. (uniform priors)
- For 1789–1924: $y_2 = 15$, $n_2 = 30$, so posterior is Beta$(16, 16)$ and mode $\theta_2 = .5161$. (uniform priors)
- Drew 10000 samples from posterior to get distributions of θ_1s and θ_2s.
- $Pr(\theta_1 > \theta_2) = .9369$.
- Because of simulation can could be a bit different every time, but basic result is the same.
- Before going through the R code, a look at the simulated distributions.

Monte Carlo Distributions 1e4



How to do this in R

- Find posteriors for both groups
- Take many random draws from each posterior:

```
# Monte carlo
S= 1e4
theta1 ← rbeta(S, a.new, b.new)
theta2 ← rbeta(S, a.old, b.old)
```

- Tally up

```
# Find proportion of times that theta1 > theta2
mean(theta1>theta2)
# Another way...
new.greater ← ifelse(theta1>theta2,1,0)
table(new.greater)
```

COVID19

We might want to know whether the probability of death increased over time. The "population" is the same in that cases are from US, but from different time points; that is, one population is individuals who had COVID19 in February and the other population is individuals who had COVID19 in March 2000.

Back to the Rmarkdown file.

Practice

Return the problem about drop out due to tuition increase. Suppose that at the Iowa State University a random sample of 50 students were asked whether they would have to quit school if turion increased by 19%. Of the 50, 10 said they would have to drop out. Is the proportion at Iowa State larger than that at University of Iowa?

What We Covered

- Beta and Binomial Distributions.
- Conjugate prior.
- Exchangeability.
- Inference for a proportion.
- Credible intervals & High Density intervals.
- Predicting new observation.
- Up-dating posterior distribution given new data.
- How Learning occurs.
- A little model checking.

What We Covered

- Monte Carlo
- Comparing two proportions