Serial Correlation

Edps/Psych/Soc 587

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Model for Level 1 Residuals

There are three sources of possible variance

- Between individuals, modeled by $\boldsymbol{Z}_i\boldsymbol{U}_i$ where $\boldsymbol{U}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{T})$.
- Within individuals, $R_{it} = e_{(1)it} + e_{(2)it}$
  - Random (measurement error, variables not included, etc.), $e_{(1)it} \sim \mathcal{N}(0, \sigma^2)$.
  - Autocorrelated errors, $e_{(2)it}$... need model for this.

Autocorrelated Errors: Mini-Outline

- Possible models for R_{it}
- Back to Riesby data.
- Models for $\boldsymbol{U}_i$ and R_{it}.
- Time varying explanatory variables: Riesby — see text by Hedeker & Gibbons (2006).
- Detecting serial correlation.

Possible models for R_{it}

In an HLM/linear mixed model,

$$\underset{(r_i \times 1)}{\boldsymbol{Y}_i} = \underset{(r_i \times p)}{\boldsymbol{X}_i} \underset{(p \times 1)}{\boldsymbol{\Gamma}} + \underset{(r_i \times q)}{\boldsymbol{Z}_i} \underset{(q \times 1)}{\boldsymbol{U}_i} + \underset{(r_i \times 1)}{\boldsymbol{R}_i}$$

where $i = 1, \ldots, N$ and $r_i =$ number of time points for individual i.

- $\boldsymbol{U}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{T})$.
- $\boldsymbol{R}_i \sim \mathcal{N}(\mathbf{0}, \sigma^2 \boldsymbol{\Omega}_i)$
- var$(\boldsymbol{Y}_i) = \boldsymbol{V}_i = \boldsymbol{Z}_i \boldsymbol{T} \boldsymbol{Z}_i' + \sigma^2 \boldsymbol{\Omega}_i$.

Possible Models for Serial Correlation

- Autoregressive (AR).
- Moving average (MA).
- Autoregressive with a moving average (ARMA).
- Toeplitz.
- Others.

Autoregressive Errors

First order autoregressive process, AR1:

$$
\begin{array}{lrcl}
\text{Time 1:} & R_{i1} & = & \epsilon_{i1} \\
\text{Time 2:} & R_{i2} & = & \rho R_{i1} + \epsilon_{i2} \\
 & & = & \rho \epsilon_{i1} + \epsilon_{i2} \\
\text{Time 3:} & R_{i3} & = & \rho R_{i2} + \epsilon_{i3} \\
 & & = & \rho(\rho \epsilon_{i1} + \epsilon_{i2}) + \epsilon_{i3} \\
 & \vdots & & \vdots \\
\text{Time t:} & R_{it} & = & \rho R_{i(t-1)} + \epsilon_{it}
\end{array}
$$

Autoregressive Errors

First order autoregressive process, AR1:

$$R_{it} = \rho R_{i(t-1)} + \epsilon_{it}$$

where

- $\epsilon_{it} \sim \mathcal{N}(0, \sigma^2_\epsilon)$ i.i.d.
- ρ is autocorrelation coefficient, $0 \leq |\rho| < 1$.
- Stationarity: variance of R_{it} and covariance between R_{it} and $R_{it'}$ are independent of t.
- Resulting error variance structure. . .

Autoregressive Errors

$$\sigma^2\mathbf{\Omega} = \frac{\sigma_\epsilon^2}{(1-\rho^2)}\begin{pmatrix} 1 & \rho & \rho^2 & \dots & \rho^{r_i-1} \\ \rho & 1 & \rho & \dots & \rho^{r_i-2} \\ \rho^2 & \rho & 1 & \dots & \rho^{r_i-3} \\ \vdots & \vdots & \vdots & \ddots & \dots \\ \rho^{r_i-1} & \rho^{r_i-2} & \rho^{r_i-3} & \dots & 1 \end{pmatrix}.$$

In SAS/MIXED and nlme R package, AR1 is parameterized as

$$\sigma^{2*}\mathbf{\Omega} = \sigma_\epsilon^{2*}\begin{pmatrix} 1 & \rho & \rho^2 & \dots & \rho^{r_i-1} \\ \rho & 1 & \rho & \dots & \rho^{r_i-2} \\ \rho^2 & \rho & 1 & \dots & \rho^{r_i-3} \\ \vdots & \vdots & \vdots & \ddots & \dots \\ \rho^{r_i-1} & \rho^{r_i-2} & \rho^{r_i-3} & \dots & 1 \end{pmatrix},$$

where $\sigma_\epsilon^{2*} = \sigma_\epsilon^2/(1-\rho^2)$.

Notes Regarding AR1

- AR1 process is a regression equation in which R_{it} depends on it's past values.
- Since R_{it} only depends on it's past values, this is a Markov process.
- $\mathbf{\Omega}$ is defined by ρ, the autocorrelation coefficient.
- Non-stationarity: If the variance of R_{it} and the covariance between R_{it} and $R_{it'}$ increases over time, then you have non-stationarity.
  - Not available in SAS.
  - Available in Hedekker's MIXREG program.
  - Not readily available in R (that I know of).

Simulated Data: No Serial Correlation

Random Intercept and Random Slope for Time:

$$Y_{it} = 10 + (\text{time})_{it} + U_{0j} + U_{1j}(\text{time})_{it} + R_{it}$$

where

- (time)$= t = 1\ldots,20$, and $N = 50$ individuals.
- $\boldsymbol{U}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{T})$ with

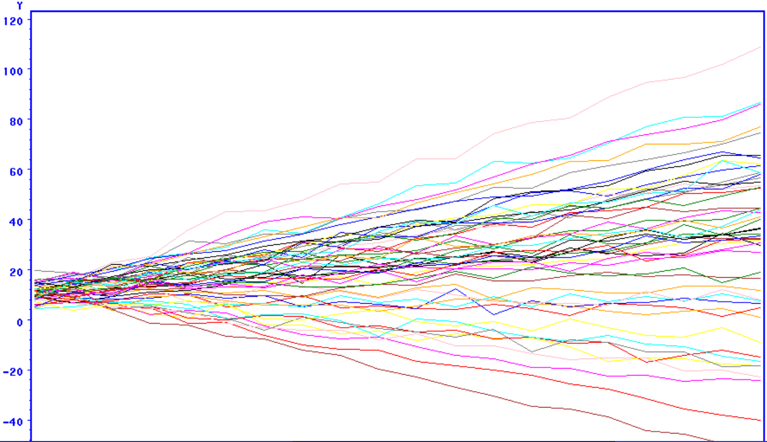
$$\boldsymbol{T} = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$$

- $R_{it} \sim \mathcal{N}(\mathbf{0}, 4)$.

No Serial Correlation: Data

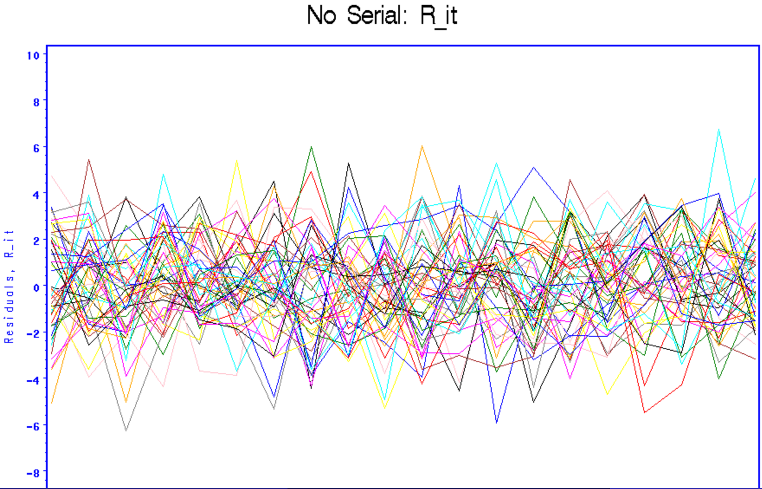
$R_{it} \sim \mathcal{N}(0, \sigma^2)$ iid.

No Serial: Data for 50 Simulated Individuals



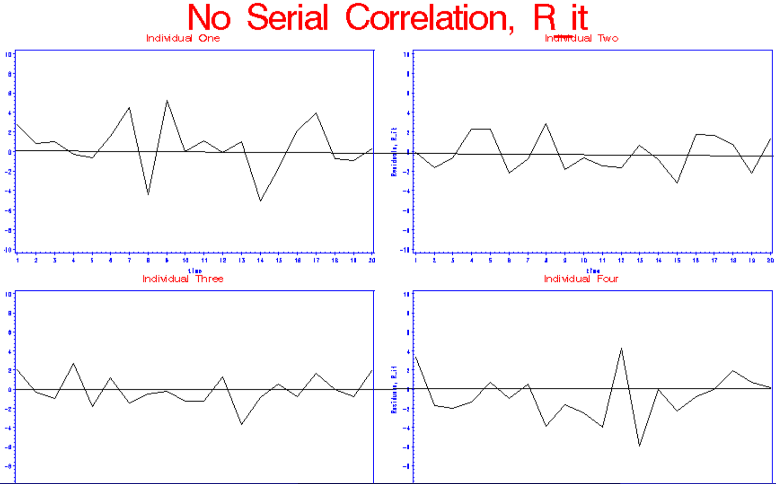
No Serial Correlation: R_{it}

$R_{it} \sim \mathcal{N}(0, \sigma^2)$ iid.



No Serial Correlation: R_{it}

$R_{it} \sim \mathcal{N}(0, \sigma^2)$ iid.



Simulated AR1 Data

Random intercept and random slope:

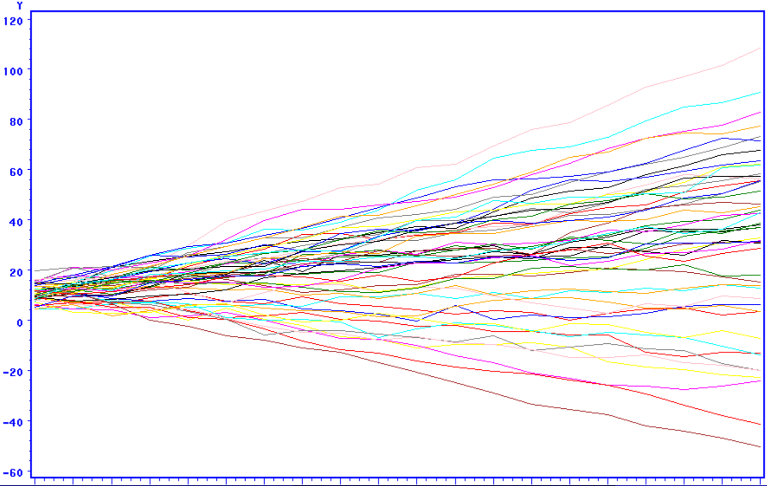
$$Y_{it} = 10 + (\text{time})_{it} + U_{0j} + U_{1j}(\text{time})_{it} + .75R_{i(t-1)} + \epsilon_{it}$$

where

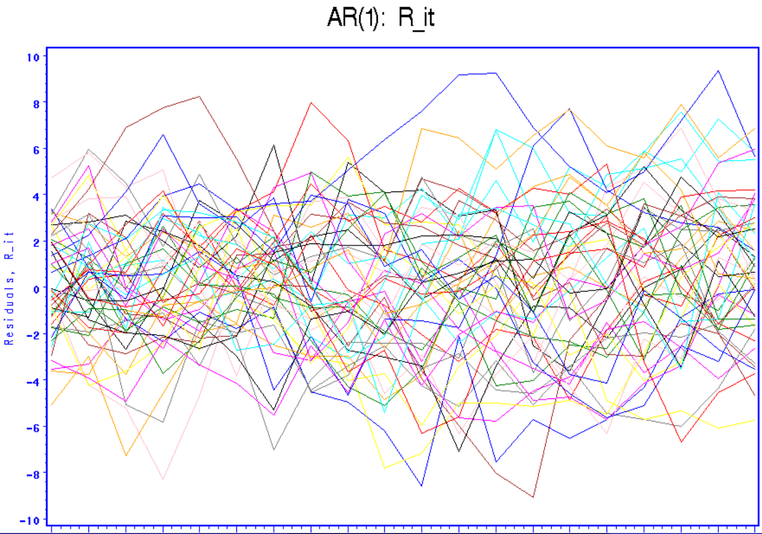
- (time)$= t = 1\ldots, 20$, and $N = 50$ individuals.
- $\boldsymbol{U}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{T})$ with
$$\boldsymbol{T} = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$$
- $\epsilon_{it} \sim \mathcal{N}(\mathbf{0}, 4)$.
- $R_{it} = .75R_{i(t-1)} + \epsilon_{it}$.

Eg of AR1: The Data

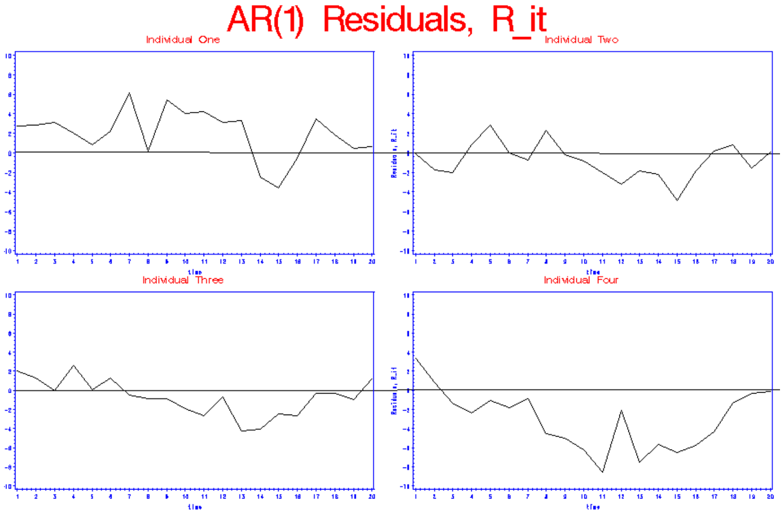
AR(1): Data for 50 Simulated Individuals



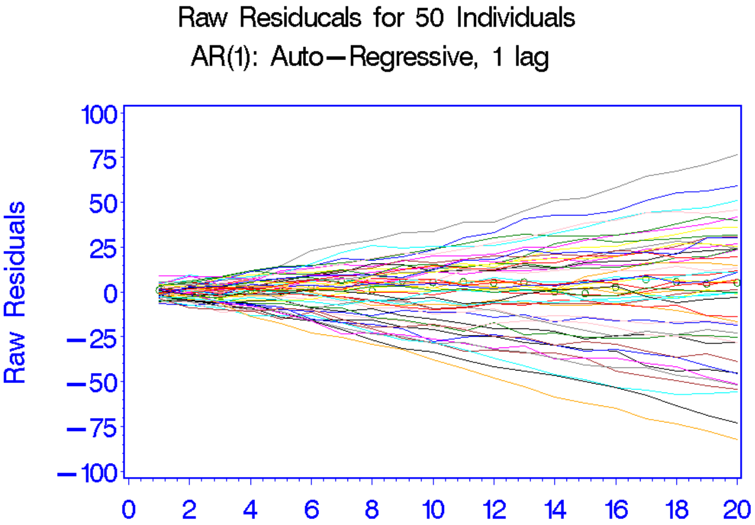
Eg of AR1: The R_{it}



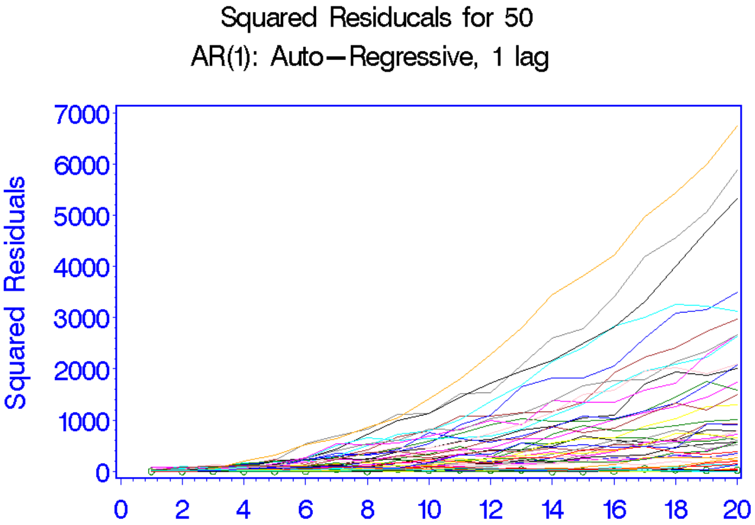
Eg of AR1: Some R_{it}



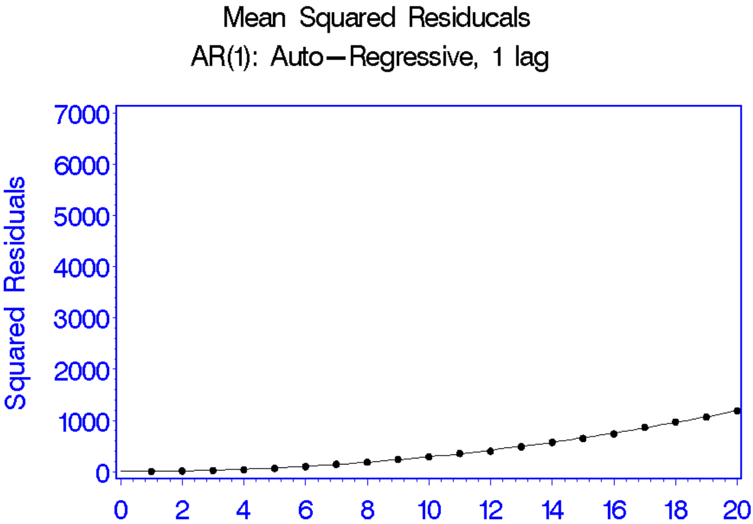
Eg of AR1: OLS $\hat{R}_{it}$



Eg of AR1: OLS $\hat{R}^2_{it}$



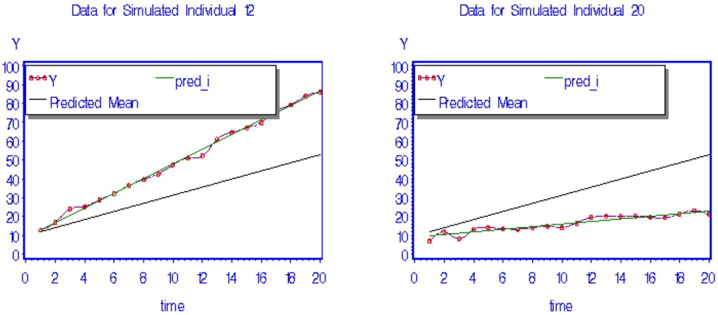
Eg of AR1: OLS mean $\hat{R}^2_{it}$



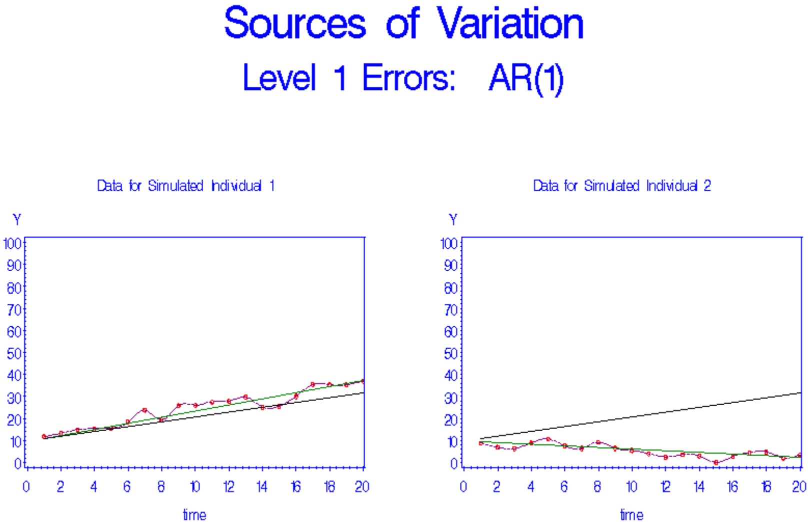
Example of AR1 (continued)

Sources of Variation

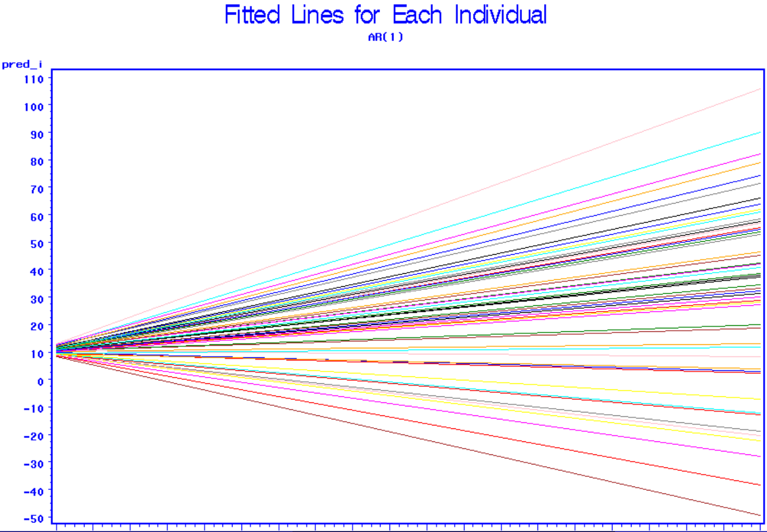
Level 1 Errors: AR(1)



Example of AR1 (continued)



Example of AR1 (continued)



SAS/MIXED and AR1 (continued)

```
PROC MIXED data=new1 method=ml;
CLASS i occasion;
MODEL y= time / solution ;
RANDOM intercept time /type=un subject=i solution G;
REPEATED occasion / subject=i type=AR(1) R;
```

- REPEATED works much the same way the RANDOM does.
- Need two time variables (one continuous/numerical & one classification).

R package nlme code given later for Reisby depression example.

SAS/MIXED and AR1 (continued)

Covariance Parameter Estimates

Cov Parm		Subject	Actual	Estimate
τ_{00}	UN(1,1)	i	4.00	0.94
τ_{10}	UN(2,1)	i	-2.00	-1.01
τ_{11}	UN(2,2)	i	4.00	3.04
ρ	AR(1)	i	0.75	0.72
σ_{ϵ}^{2*}	Residual		9.14	8.15

Note: $\hat{\sigma}^2 = \hat{\sigma}^{2*}(1 - \hat{\rho}^2) = 8.15(1 - .75^2) = 3.93$. The value used to simulate data was 4.

SAS/MIXED and AR1 (continued)

Estimated $\hat{\sigma}^{2*}\hat{\Omega}$

r Row	Col1	Col2	Col3	Col4	Col5	Col6	Col7	Col8
1	8.1509							
2	5.8668	8.1509						
3	4.2228	5.8668	8.1509					
4	3.0395	4.2228	5.8668	8.1509				
5	2.1878	3.0395	4.2228	5.8668	8.1509			
6	1.5747	2.1878	3.0395	4.2228	5.8668	8.1509		
7	1.1334	1.5747	2.1878	3.0395	4.2228	5.8668	8.1509	
r 8	0.8158	1.1334	1.5747	2.1878	3.0395	4.2228	5.8668	8.1509
9	0.5872	0.8158	1.1334	1.5747	2.1878	3.0395	4.2228	5.8668
10	0.4227	0.5872	0.8158	1.1334	1.5747	2.1878	3.0395	4.2228
11	0.3042	0.4227	0.5872	0.8158	1.1334	1.5747	2.1878	3.0395
12	0.2190	0.3042	0.4227	0.5872	0.8158	1.1334	1.5747	2.1878
13	0.1576	0.2190	0.3042	0.4227	0.5872	0.8158	1.1334	1.5747
14	0.1134	0.1576	0.2190	0.3042	0.4227	0.5872	0.8158	1.1334
15	0.08166	0.1134	0.1576	0.2190	0.3042	0.4227	0.5872	0.8158

First Order Moving Average

$$
\begin{array}{lrcl}
\text{Time 1:} & R_{i1} & = & \epsilon_{i1} \\
\text{Time 2:} & R_{i2} & = & \epsilon_{i2} - \theta\epsilon_{i1} \\
\text{Time 3:} & R_{i3} & = & \epsilon_{i3} - \theta\epsilon_{i2} \\
 & \vdots & & \vdots \\
\text{Time t:} & R_{it} & = & \epsilon_{it} - \theta\epsilon_{i,(t-1)}
\end{array}
$$

where

- $\epsilon_{it} \sim \mathcal{N}(0, \sigma^2_\epsilon)$ i.i.d.
- θ is the autocorrelation coefficient.

First Order Moving Average

The covariance matrix for R_{it} is

$$\sigma_\epsilon^2 \mathbf{\Omega} = \sigma_\epsilon^2 \begin{pmatrix} 1+\theta^2 & -\theta & 0 & \dots & 0 \\ -\theta & 1+\theta^2 & -\theta & \dots & 0 \\ 0 & -\theta & 1+\theta^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots \\ 0 & 0 & 0 & \dots & 1+\theta^2 \end{pmatrix}$$

.

MA1 and SAS/MIXED

SAS/MIXED doesn't estimate

$$\sigma_\epsilon^2 \boldsymbol{\Omega} = \sigma_\epsilon^2 \begin{pmatrix} 1+\theta^2 & -\theta & 0 & \dots & 0 \\ -\theta & 1+\theta^2 & -\theta & \dots & 0 \\ 0 & -\theta & 1+\theta^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots \\ 0 & 0 & 0 & \dots & 1+\theta^2 \end{pmatrix}$$

The closest you can come to this is in SAS is TYPE=TOEP(2),

$$\text{cov}(\boldsymbol{R}_i) = \begin{pmatrix} \sigma^2 & \sigma_1 & 0 & \dots & 0 \\ \sigma_1 & \sigma^2 & \sigma_1 & \dots & 0 \\ 0 & \sigma_1 & \sigma^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \dots \\ 0 & 0 & 0 & \dots & \sigma^2 \end{pmatrix}$$

.

Simulated MA1 Data

$$Y_{it} = 10 + (\text{time})_{it} + U_{0j} + U_{1j}(\text{time})_{it} + \epsilon_{it} - .75\epsilon_{i,(t-1)}$$

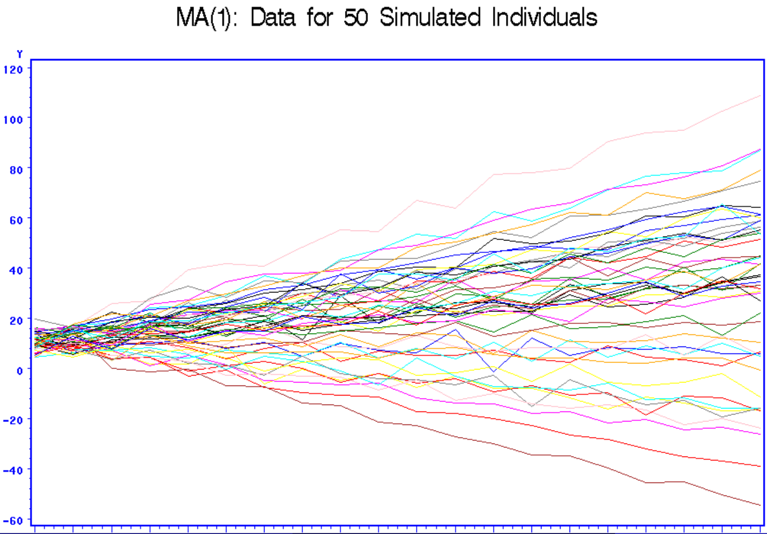
where

- (time)$= t = 1\ldots, 20$, and $N = 50$ individuals.
- $\boldsymbol{U}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{T})$ with

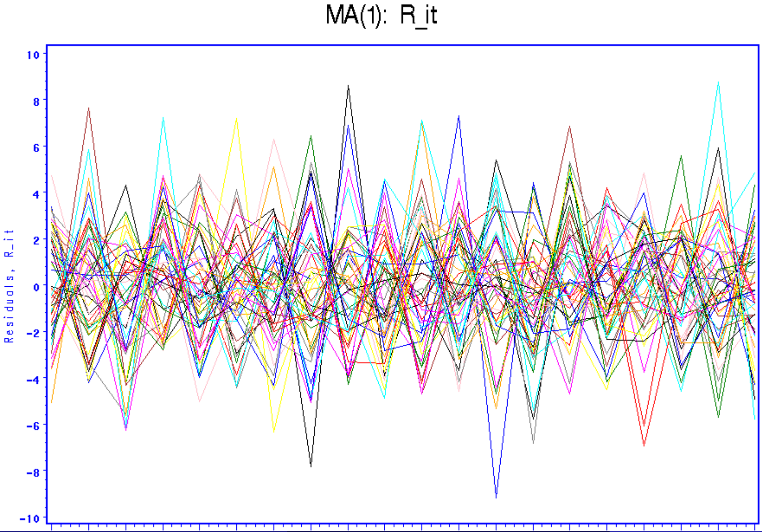
$$\boldsymbol{T} = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$$

- $\epsilon_{it} \sim \mathcal{N}(\mathbf{0}, 4)$.
- $R_{it} = \epsilon_{it} - .75\epsilon_{i,(t-1)}$.

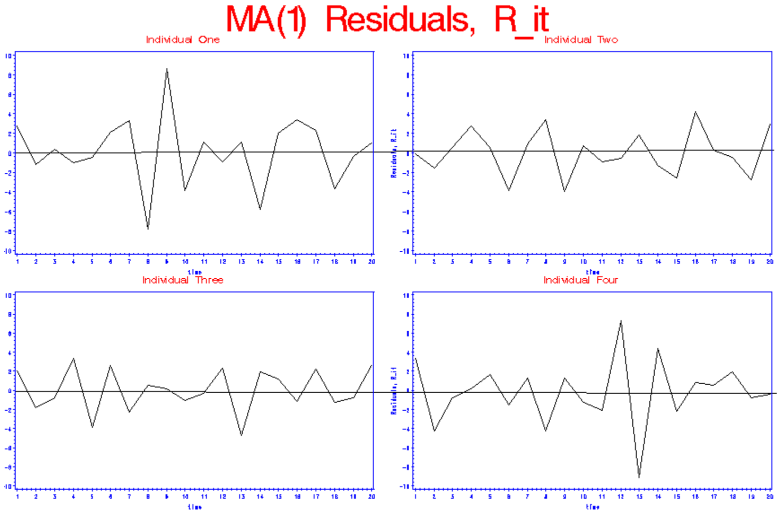
Eg of MA1 — Data



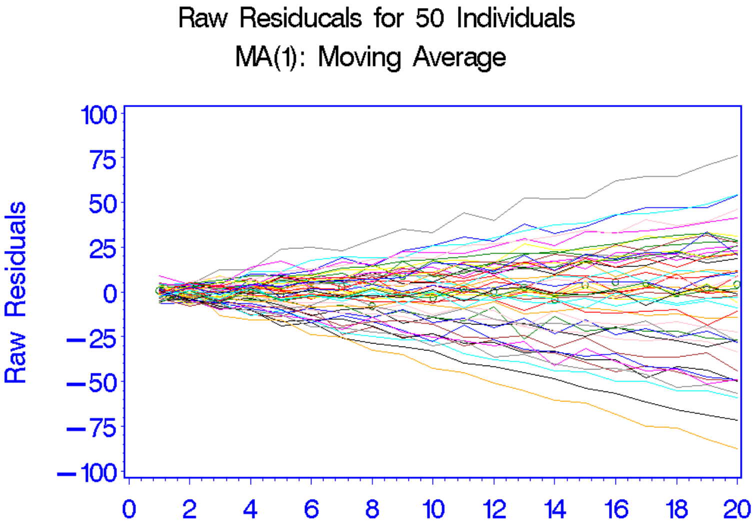
Eg of MA1 — R_{it}



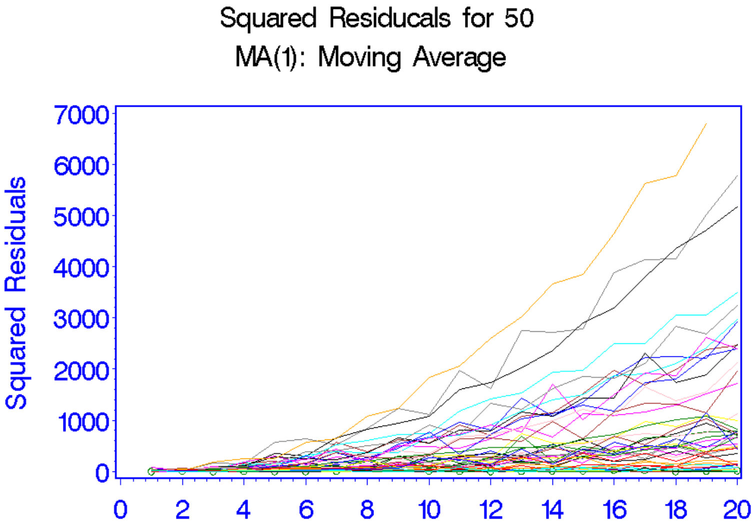
Eg of MA1 — R_{it}


MA(1) Residuals, R_it
Individual One
Individual Two
Individual Three
Individual Four

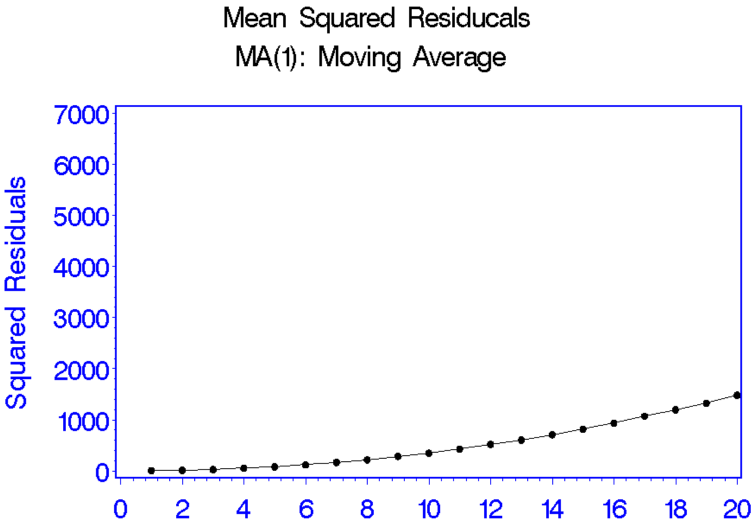
Eg of MA1 — OLS $\hat{R}_{it}$



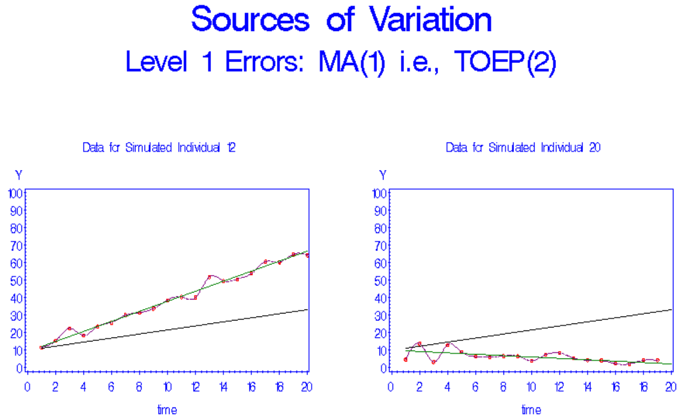
Eg of MA1 — OLS $\hat{R}_{it}^2$



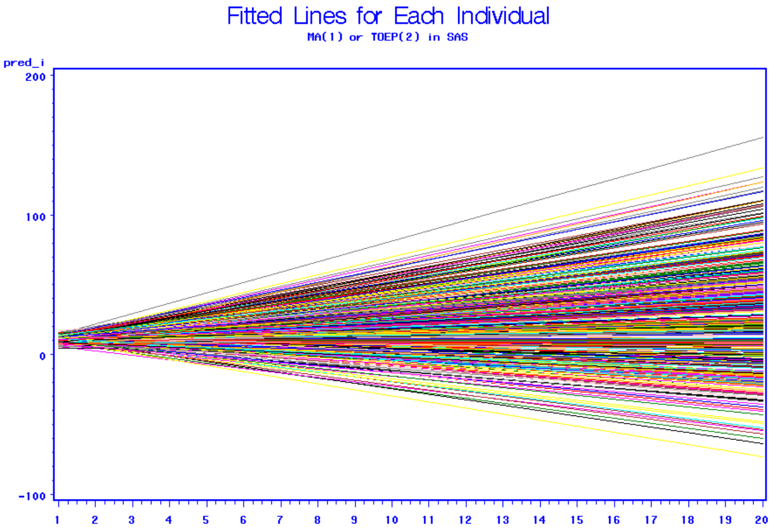
Eg of MA1 — Mean Sq Residuals



Example of MA1 (continued)



Example of MA1 (continued)



SAS/MIXED and TOEP(2) (continued)

Covariance Parameter Estimates

Cov Parm	Subject	Estimate
UN(1,1)	i	4.0533
UN(2,1)	i	-1.3121
UN(2,2)	i	3.0624
TOEP(2)	i	-2.9952
Residual		6.2266

SAS/MIXED and TOEP(2): $\hat{\sigma}^{2*}\hat{\Omega}$

Row	Col1	Col2	Col3	Col4	Col5	Col6	Col7	Co
1	6.2266	-2.9952						
2	-2.9952	6.2266	-2.9952					
3		-2.9952	6.2266	-2.9952				
4			-2.9952	6.2266	-2.9952			
5				-2.9952	6.2266	-2.9952		
6					-2.9952	6.2266	-2.9952	
7						-2.9952	6.2266	-2.995
8							-2.9952	6.226
9								-2.995
10								
11								
12								

Autoregressive-Moving Average

"ARMA(1,1)"

$$R_{it} = \rho R_{i,t-1} + \epsilon_{it} - \theta \epsilon_{i,(t-1)}$$

where

- $\epsilon_{it} \sim \mathcal{N}(0, \sigma^2_\epsilon)$ iid.
- ρ is autocorrelation coefficient, $|\rho| < 1$.
- Stationarity: variance of R_{it} and covariance between R_{it} and $R_{it'}$ are independent of t.
- θ is the autocorrelation coefficient.

ARMA(1,1) Error Covariance Matrix

$$\text{cov}(\boldsymbol{R}_i) = \frac{\sigma_\epsilon^2}{(1-\rho^2)} \begin{pmatrix} \xi_0 & \xi_1 & \rho\xi_1 & \ldots & \rho^{r-2}\xi_1 \\ \xi_1 & \xi_0 & \xi_1 & \ldots & \rho^{r-3}\xi \\ \rho\xi_1 & \xi_1 & \xi_0 & \ldots & \rho^{r-4}\xi_1 \\ \rho^2\xi_1 & \rho\xi_1 & \xi_1 & \ldots & \rho^{(r-5)}\xi_1 \\ \vdots & \vdots & \vdots & \ddots & \ldots \\ \rho^{(r-2)}\xi_1 & \rho^{(r-3)}\xi_1 & \rho^2(r-4)\xi_1 & \ldots & \xi_0 \end{pmatrix}$$

where

- $\xi_0 = 1 + \theta^2 - 2\rho\theta$ and $\xi_1 = (1-\rho\theta)(\rho-\theta)$.
- The MA1 term, θ, changes the lag-1 autocorrelation and then autocorrelations deceases as in AR1.

ARMA(1,1) SAS Parameterization

$$\text{cov}(\boldsymbol{R}_i) = \sigma^{2*} \begin{pmatrix} 1 & \xi^* & \rho\xi^* & \dots & \rho^{r-2}\xi^* \\ \xi^* & 1 & \xi^* & \dots & \rho^{r-3}\xi^* \\ \rho\xi^* & \xi^* & 1 & \dots & \rho^{r-4}\xi^* \\ \rho^2\xi^* & \rho\xi^* & \xi^* & \dots & \rho^{(r-5)}\xi^* \\ \vdots & \vdots & \vdots & \ddots & \dots \\ \rho^{(r-2)}\xi^* & \rho^{(r-3)}\xi^* & \rho^2(r-4)\xi^* & \dots & 1 \end{pmatrix}$$

where

- $\sigma^{2*} = (\sigma_\epsilon^2/(1-\rho^2))\xi_0 = (\sigma_\epsilon^2/(1-\rho^2))(1+\theta^2-2\rho\theta)$
- $\xi^* = (\sigma_\epsilon^2/(1-\rho^2))(1-\rho\theta)(\rho-\theta)$.

Simulated ARMA(1,1) Data

$$Y_{it} = 10 + (\text{time})_{it} + U_{0j} + U_{1j}(\text{time})_{it} + R_{it}$$

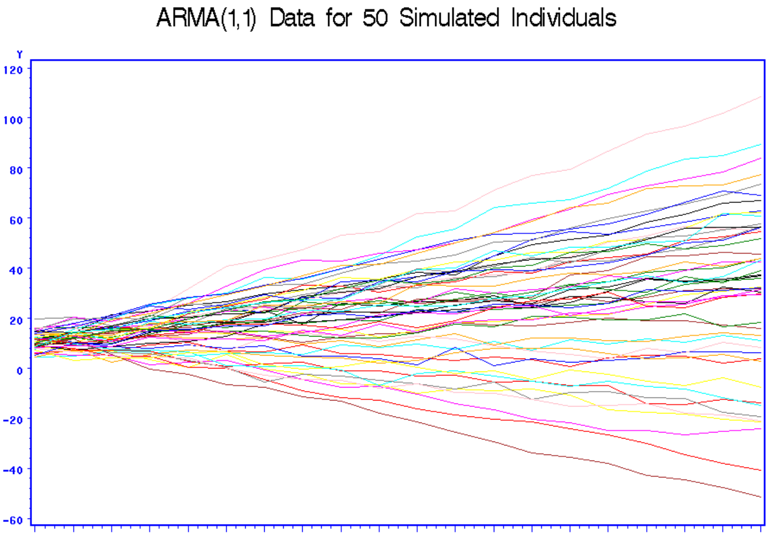
where

- $R_{it} = \epsilon_{it} + .75R_{i,t-1} - .25\epsilon_{it}$
- (time)$= t = 1\ldots,20$, and $N = 500$ individuals.
- $\boldsymbol{U}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{T})$ with

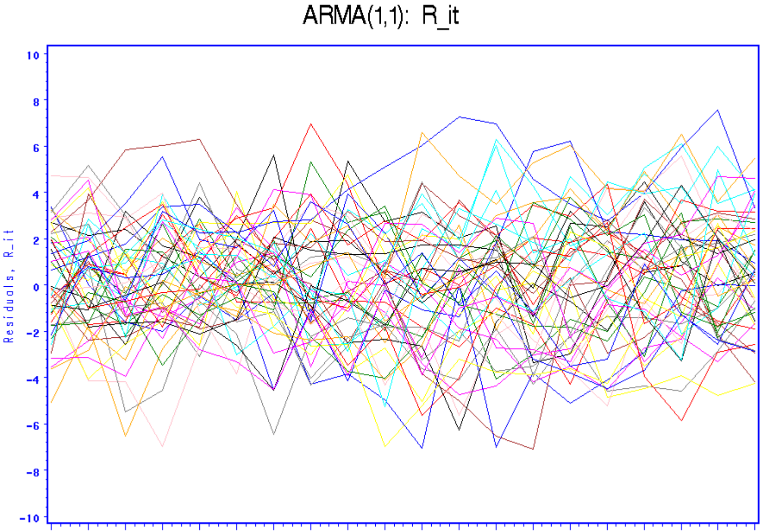
$$\boldsymbol{T} = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}$$

- $\epsilon_{it} \sim \mathcal{N}(\mathbf{0}, 4)$.

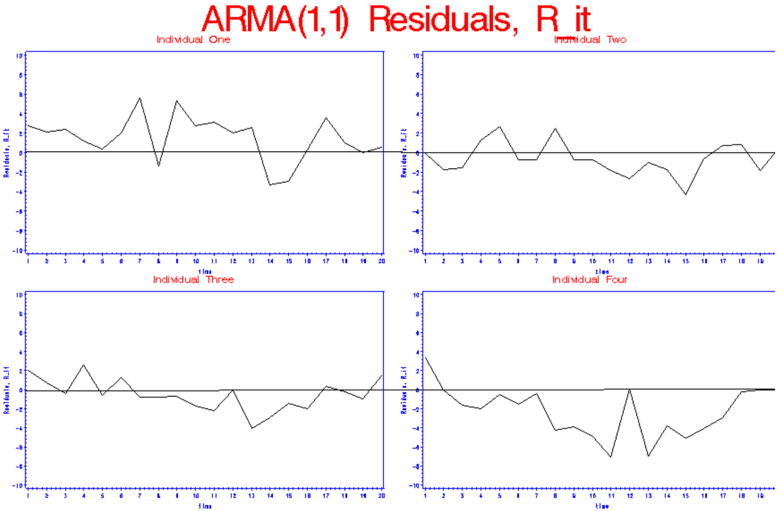
Eg of ARMA(1,1): The data



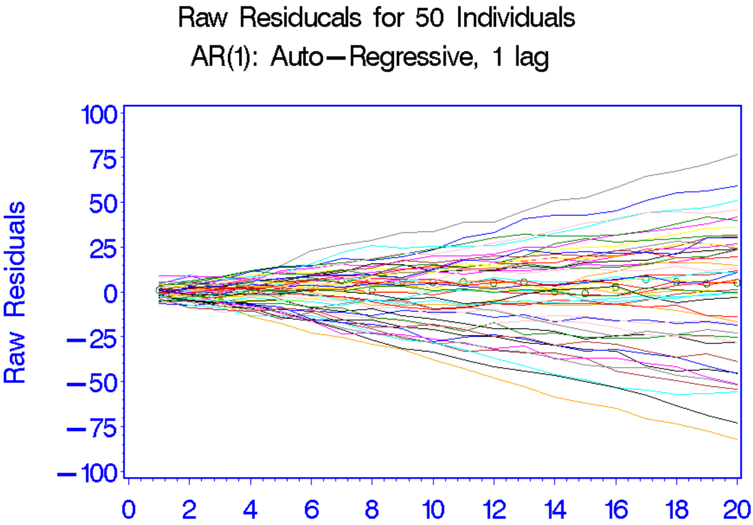
Eg of ARMA(1,1): The R_{it}



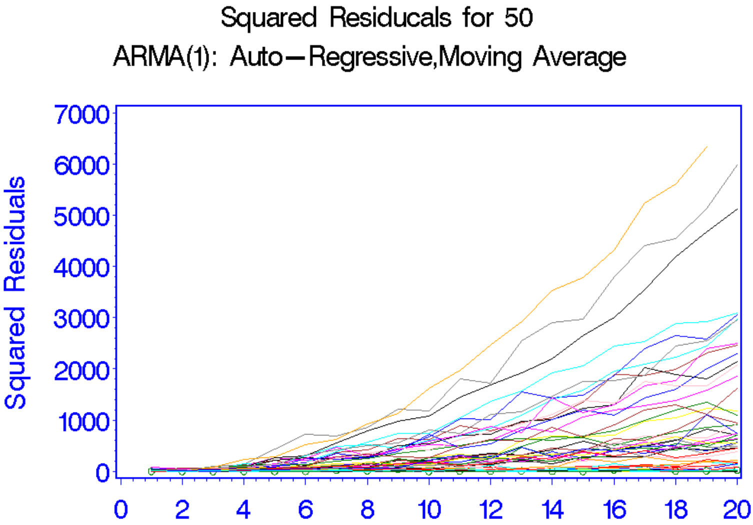
Eg of ARMA(1,1): The R_{it}



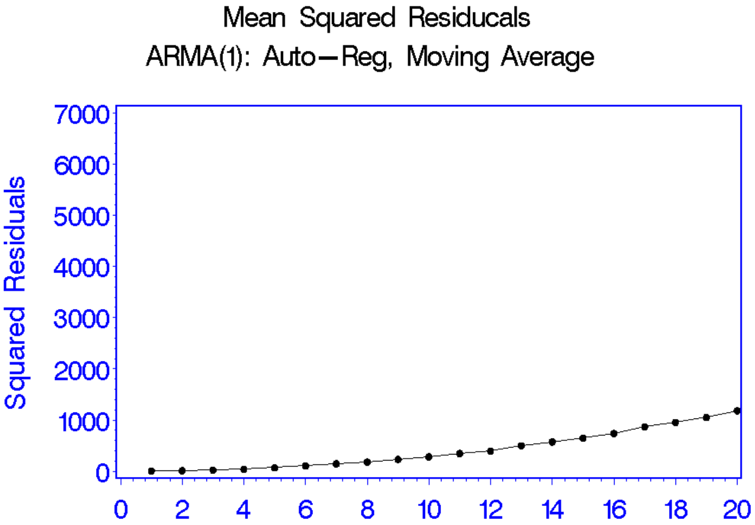
Eg of ARMA(1,1): The OLS $\hat{R}_{it}$



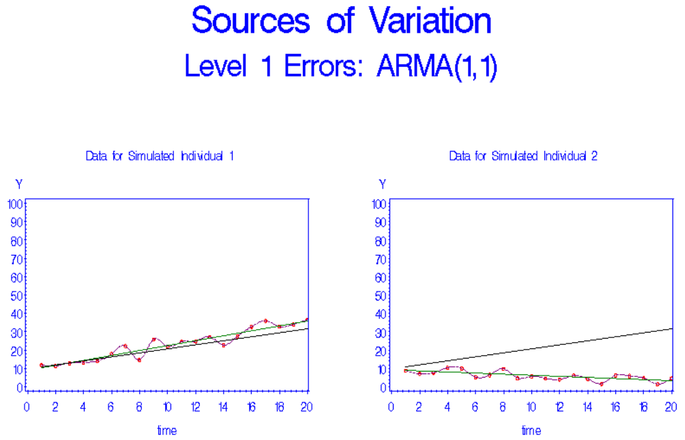
Eg of ARMA(1,1): The OLS $\hat{R}_{it}^2$



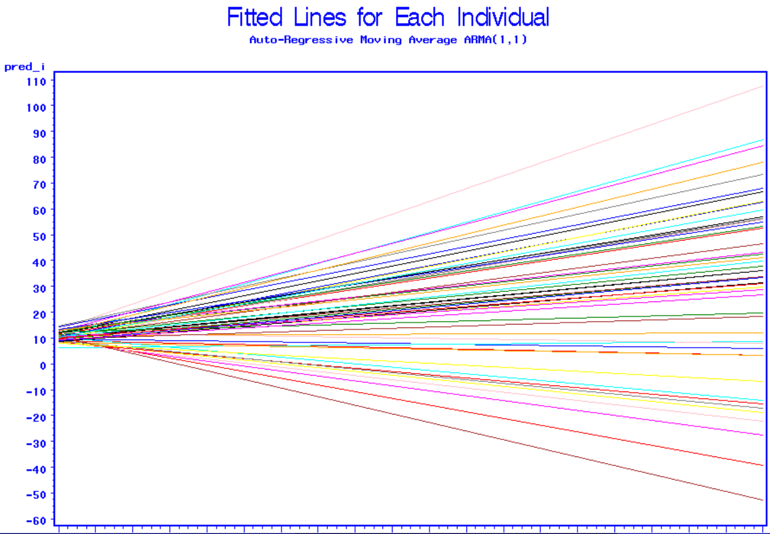
Eg of ARMA(1,1): The OLS $\hat{R}_{it}^2$



Example of ARMA(1,1) (continued)



Example of ARMA(1,1) (continued)



SAS/MIXED and ARMA(1,1)

Covariance Parameter Estimates

Cov Parm		Subject	Estimate
τ_{00}	UN(1,1)	i	2.5978
τ_{10}	UN(2,1)	i	-1.1575
τ_{11}	UN(2,2)	i	3.0521
ρ	Rho	i	.6902
ξ^*	Gamma	i	.5358
Residual			5.5804

SAS/MIXED and ARMA(1,1) (continued)

Covariance Matrix for R_{it}:

1 5.5804
2 2.9901 5.5804
3 2.0637 2.9901 5.5804
4 1.4244 2.0637 2.9901 5.5804
5 0.9831 1.4244 2.0637 2.9901 5.5804
6 0.6786 0.9831 1.4244 2.0637 2.9901 5.5804
7 0.4683 0.6786 0.9831 1.4244 2.0637 2.9901 5.5804
8 0.3233 0.4683 0.6786 0.9831 1.4244 2.0637 2.9901
9 0.2231 0.3233 0.4683 0.6786 0.9831 1.4244 2.0637 2.9901 5.5804
10 0.1540 0.2231 0.3233 0.4683 0.6786 0.9831 1.4244 2.0637 2.9901
11 0.1063 0.1540 0.2231 0.3233 0.4683 0.6786 0.9831 1.4244 2.0637
12 0.07336 0.1063 0.1540 0.2231 0.3233 0.4683 0.6786 0.9831 1.4244
13 0.05063 0.07336 0.1063 0.1540 0.2231 0.3233 0.4683 0.6786 0.9831
14 0.03495 0.05063 0.07336 0.1063 0.1540 0.2231 0.3233 0.4683 0.6786
15 0.02412 0.03495 0.05063 0.07336 0.1063 0.1540 0.2231 0.3233 0.4683
16 0.01665 0.02412 0.03495 0.05063 0.07336 0.1063 0.1540 0.2231 0.3233

Other Error Structures

- Autocorrelations of each lag are functionally related:
  - AR(1), MA(1), ARMA(1,1)
  - Gaussian
  - Fractional Polynomials
- Autocorrelations of each lag are not functionally related: Toeplitz Errors

Toeplitz Errors

General Toeplitz matrix:

$$\boldsymbol{\Sigma}_R = \sigma^2 \boldsymbol{\Omega} = \sigma^2 \begin{pmatrix} 1 & \rho_1 & \rho_2 & \rho_3 & \dots & \rho_{r-1} \\ \rho_1 & 1 & \rho_1 & \rho_2 & \dots & \rho_{r-2} \\ \rho_2 & \rho_1 & 1 & \rho_1 & \dots & \rho_{r-3} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{r-1} & \rho_{r-2} & \rho_{r-3} & \rho_{r-4} & \dots & 1 \end{pmatrix}.$$

- Higher-order lags possible, but usually assumed to be 0.
- MA(1), which has lag of 1, is Toeplitz(2).
- In SAS TYPE=TOEP($\#$ *lags* $+1$)

Back to Riesby Data

Recall that we decided on the following model:

- Level 1 model:

$$\text{HamD}_{it} = \beta_{0i} + \beta_{1i}(\text{time})_{it} + \beta_{2i}(\text{time})^2_{it} + R_{it}$$

  where $R_{it} \sim \mathcal{N}(0, \sigma^2)$ i.i.d & $\text{cov}(\boldsymbol{R}_i) = \sigma^2 \boldsymbol{I}$.

- Level 2 model:

$$\begin{aligned}
\beta_{0i} &= \gamma_{00} + \gamma_{01}(\text{Endog})_i + U_{0i} \\
\beta_{1i} &= \gamma_{10} + U_{1i} \\
\beta_{2i} &= \gamma_{20} + U_{2i}
\end{aligned}$$

  where $\boldsymbol{U}_i \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{T})$.

New Models for Riesby Data

- Linear Mixed Model:

$$\begin{aligned}\text{HamD}_{it} &= \gamma_{00} + \gamma_{10}(\text{time})_{it} + \gamma_{20}(\text{time})_{it}^2 + \gamma_{01}(\text{Endog})_i \\ &\quad + U_{0i} + U_{1i}(\text{time})_{it} + U_{2i}(\text{time})_{it}^2 + R_{it}\end{aligned}$$

- Model for $\text{cov}(\boldsymbol{R}_i)$
  - AR(1)
  - MA(1)
  - ARMA(1,1)

Global Fit Statistics

Model	-2LnLike	AIC	BIC
Empty/Null	2501.1	2507.1	2513.7
Preliminary HLM	2204.0	2228.0	2254.3
No "endog*week"	2204.0	2226.0	2250.1
No "endog*week" and no "endog"	2207.6	2227.6	2249.5
AR(1)	2203.1	2227.1	2253.4
TOEP(2)	2203.3	2227.3	2253.6
ARMA(1,1)	*Estimated G matrix is not positive definite*		

Cov. Parameter Estimates: AR(1)

Cov Parm	Estimate	Standard Error	Z Value	Pr Z
UN(1,1)	7.45	4.80	1.55	0.06
UN(2,1)	0.10	2.90	0.04	0.97
UN(2,2)	4.85	3.54	1.37	0.09
UN(3,1)	−0.18	0.45 -	0.41	0.68
UN(3,2)	−0.65	0.60 -	1.08	0.28
UN(3,3)	0.14	0.11	1.23	0.11
AR(1)	0.15	0.17	0.88	0.39
Residual	12.41	2.99	4.14	$< .00$

Estimated cov($\boldsymbol{R}_i$): AR(1)

Row	Col1	Col2	Col3	Col4	Col5	Col6
1	12.417	1.878	0.284	0.042	0.006	0.000
2	1.878	12.417	1.878	0.284	0.042	0.006
3	0.284	1.878	12.417	1.878	0.284	0.042
4	0.042	0.284	1.878	12.417	1.878	0.28
5	0.006	0.042	0.284	1.878	12.417	1.87
6	0.000	0.006	0.042	0.284	1.878	12.41

Note: $(2, 1)$ covariance equals $(.15)(12.417) = 1.878$.
$(3, 1)$ covariance equals $(.15)^2(12.417) = .284$.

Cov. Parameter Estimates: TOEP(2)

Cov Parm	Estimate	Standard Error	Z Value	Pr Z
UN(1,1)	8.36	3.98	2.10	0.02
UN(2,1)	−0.32	2.65	−0.12	0.90
UN(2,2)	5.44	3.13	1.74	0.04
UN(3,1)	−0.15	0.44	−0.33	0.73
UN(3,2)	−0.75	0.55	−1.37	0.17
UN(3,3)	0.16	0.10	1.52	0.06
TOEP(2)	1.18	1.41	0.83	0.41
Residual	11.66	1.89	6.16	$< .00$

Estimated cov($\boldsymbol{R}_i$): TOEP(1)

week	week 1	week 2	week 3	week 4	week 5	week 6
1	11.66	1.18	0	0	0	0
2	1.18	11.66	1.18	0	0	0
3	0	1.18	11.66	1.18	0	0
4	0	0	1.18	11.66	1.18	0
5	0	0	0	1.18	11.66	1.18
6	0	0	0	0	1.18	11.66

Testing Random Quadratic Term, U_{2i}

Model	-2LnLike	AIC	BIC
Empty/Null	2501.1	2507.1	2513.7
Preliminary HLM	2204.0	2228.0	2254.3
No "endog*week"	2204.0	2226.0	2250.1
AR(1)	2203.1	2227.1	2253.4
TOEP(2)	2203.3	2227.3	2253.6
Only U_{0i} and U_{1i} (no U_{2i})			
AR(1)	2206.5	2224.5	2244.2
TOEP(2)	2209.0	2227.0	2246.7
ARMA(1,1)	*Estimated G matrix is not positive definite*		

Dropping Random Quadratic, U_{2i}

- Hypothesis Test:

$$H_o : \tau_2^2 = \tau_{12} = \tau_{20} = 0 \quad \text{versus} \quad H_a : \quad \text{Not} \quad H_o.$$

- Test statistic $= (2206.5 - 2203.1) = 3.4.$
- p-value equals a mixture of χ_3^2 and χ_2^2:

$$p\text{-value} = \frac{1}{2}(.334 + .183) = .258$$

- Retain H_o; drop quadratic random effect.

Next Steps?

- Remove fixed effect for the quadratic term:

$$H_o : \gamma_{20} = 0 \quad \text{vs} \quad H_a : \gamma_{20} \neq 0.$$

$t = 0.55$, Satter. $df = 87.2 \longrightarrow p = 0.58$.

- Global fits

Model	-2LnLike	AIC	BIC
Empty/Null	2501.1	2507.1	2513.7
Preliminary HLM	2204.0	2228.0	2254.3
No "endog*week"	2204.0	2226.0	2250.1
AR(1)	2203.1	2227.1	2253.4
AR(1) w/o U_{2i}	2206.5	2224.5	2244.2
AR(1) w/o quadratic	2206.8	2222.8	2240.4

Model Refinement (continued)

Random linear trend with $AR(1)$:
Covariance Parameter Estimates

Cov Parm	Estimate	Standard Error	Z Value	Pr Z
UN(1,1)	3.90	6.09	0.64	0.26
UN(2,1)	0.34	1.40	0.24	0.81
UN(2,2)	1.28	0.66	1.93	0.03
AR(1)	0.37	0.15	2.50	0.01
Residual	17.97	4.56	3.94	$< .01$

Dropping Random Intercept

Global fits

Model	-2LnLike	AIC	BIC	# Param
Empty/Null	2501.1	2507.1	2513.7	3
Preliminary HLM	2204.0	2228.0	2254.3	12
No "endog*week"	2204.0	2226.0	2250.1	11
AR(1)	2203.1	2227.1	2253.4	12
AR(1) w/o U_{2i}	2206.5	2224.5	2244.2	9
AR(1) w/o quadratic	2206.8	2222.8	2240.4	8
U_{1i} & AR(1) w/o U_{0i}	2209.1	2221.1	2234.2	6

Dropping Random Intercept

- Hypothesis:

$$H_o : \tau_0^2 = \tau_{01} = 0 \quad \text{versus} \quad H_a : \text{ Not } H_o$$

- Difference between -2lnLike

$$2209.1 - 2206.8 = 2.3$$

- p-value is mixture of χ_2^2 and χ_1^2,

$$p\text{–value} = \frac{1}{2}(.417 + .269) = .34$$

- Retain H_o; don't need a random intercept.
- But we keep fixed effects model for the intercept

$$\beta_{0i} = \gamma_{00} + \gamma_{01}(\text{Endog})_i$$

Parameter Estimates

Covariance Parameter Estimates

Cov Parm	Estimate	Standard Error	Z Value	Pr Z
UN(1,1)	1.45	0.43	3.40	0.0003
AR(1)	0.47	0.07	6.95	$< .0001$
Residual	21.51	2.74	7.84	$< .0001$

Solution for Fixed Effects

Effect	Estimate	Std. Error	DF	t Value	Pr ¿ —t—
Intercept	24.2905	0.6626	83.7	36.66	$< .0001$
time	-2.3259	0.2201	126	-10.57	$< .0001$
Endog$=0$	-1.8071	0.8865	92.1	-2.04	0.0444
Endog$=1$	.	.	.	.	

Final Model

- Subjective decision.
- I like simple ones, so let's look at

$$\begin{aligned} \text{HamD}_{it} &= \gamma_{00} + \gamma_{10}(\text{week})_{it} + \gamma_{01}(\text{endog})_i \\ &\quad + U_{1i}(\text{week})_{it} + \rho R_{i,(t-1)} + \epsilon_{it} \end{aligned}$$

  for $(\text{week})_{it} = 0, \ldots, 5$.
- Fixed effect structure (overall regression):

$$\widehat{\text{HamD}}_{it} = 24.29 - 2.33(\text{week})_{it} - 1.81(\text{endog})_i$$

- Covariance for $\boldsymbol{Y}_i$?

Final Model: (continued)

Estimated Covariance Matrix for $\boldsymbol{Y}_i$:

$$\boldsymbol{\Sigma}_{\boldsymbol{y}_i} = \boldsymbol{Z}_i \boldsymbol{T} \boldsymbol{Z}_i' + \sigma_\epsilon^2 \boldsymbol{\Omega}$$

where $\boldsymbol{T} = \tau_1^2$ and $\boldsymbol{Z}_i' = (0, 1, 2, 3, 4, 5)$, so

$$\boldsymbol{Z}_i \boldsymbol{T} \boldsymbol{Z}_i' = \tau_1^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 4 & 6 & 8 & 10 \\ 0 & 3 & 6 & 9 & 12 & 15 \\ 0 & 4 & 8 & 12 & 16 & 20 \\ 0 & 5 & 10 & 15 & 20 & 25 \end{pmatrix},$$

and

Covariance Matrix for $\boldsymbol{Y}_i$

Part due to autoregressive model,

$$\sigma_\epsilon^2 \boldsymbol{\Omega} = \sigma_\epsilon^2 \begin{pmatrix} 1 & \rho & \rho^2 & \rho^3 & \rho^4 & \rho^5 \\ \rho & 1 & \rho & \rho^2 & \rho^3 & \rho^4 \\ \rho^2 & \rho & 1 & \rho & \rho^2 & \rho^3 \\ \rho^3 & \rho^2 & \rho & 1 & \rho & \rho^2 \\ \rho^4 & \rho^3 & \rho^2 & \rho & 1 & \rho \\ \rho^5 & \rho^4 & \rho^3 & \rho^2 & \rho & 1 \end{pmatrix}$$

Covariance Matrix for $\boldsymbol{Y}_i$

$$
\boldsymbol{\Sigma}_{\boldsymbol{y}_i} = \begin{pmatrix}
\sigma^2 & \rho\sigma_\epsilon^2 & \rho^2\sigma_\epsilon^2 & \rho\sigma_\epsilon^3 & \rho^4\sigma_\epsilon^2 & \rho\sigma_\epsilon^5 \\
\rho\sigma_\epsilon^2 & \tau_1^2 + \sigma_\epsilon & 2\tau_1^2 + \rho\sigma_\epsilon^2 & 3\tau_1^2 + \rho^2\sigma_\epsilon^2 & 4\tau_1^2 + \rho^3\sigma_\epsilon^2 & 5\tau_1^2 + \rho^6\sigma_\epsilon^2 \\
\rho^2\sigma_\epsilon^2 & 2\tau_1^2 + \rho\sigma_\epsilon^2 & 4\tau_1^2 + \sigma_\epsilon^2 & 6\tau_1^2 + \rho\sigma_\epsilon^2 & 8\tau_1^2 + \rho^2\sigma_\epsilon^2 & 10\tau_1^2 + \rho^3\sigma_\epsilon^2 \\
\rho^3\sigma_\epsilon^2 & 3\tau_1^2 + \rho^2\sigma_\epsilon^2 & 6\tau_1^2 + \rho\sigma_\epsilon^2 & 9\tau_1^2 + \sigma_\epsilon^2 & 12\tau_1^2 + \rho\sigma_\epsilon^2 & 15\tau_1^2 + \rho^2\sigma_\epsilon^2 \\
\rho^4\sigma_\epsilon^2 & 4\tau_1^2 + \rho^3\sigma_\epsilon^2 & 8\tau_1^2 + \rho^2\sigma_\epsilon^2 & 12\tau_1^2 + \rho\sigma_\epsilon^2 & 16\tau_1^2 + \sigma_\epsilon^2 & 20\tau_1^2 + \rho\sigma_\epsilon^2 \\
\rho^5\sigma_\epsilon^2 & 5\tau_1^2 + \rho^4\sigma_\epsilon^2 & 10\tau_1^2 + \rho^3\sigma_\epsilon^2 & 15\tau_1^2 + \rho^2\sigma_\epsilon^2 & 20\tau_1^2 + \rho\sigma_\epsilon^2 & 25\tau_1^2 + \sigma_\epsilon^2
\end{pmatrix}
$$

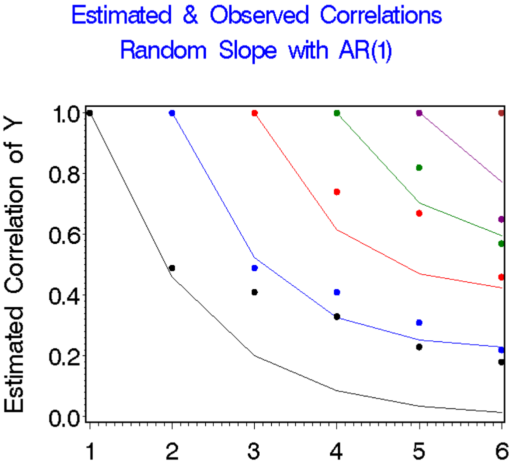
$$
\widehat{\boldsymbol{\Sigma}}_{\boldsymbol{y}_i} = \begin{pmatrix}
21.51 & 10.21 & 4.85 & 2.30 & 1.09 & 0.52 \\
10.21 & 21.51 & 10.21 & 4.85 & 2.30 & 1.19 \\
4.85 & 10.21 & 21.51 & 10.21 & 4.85 & 2.30 \\
2.30 & 4.85 & 10.21 & 21.51 & 10.21 & 4.85 \\
1.09 & 2.30 & 4.85 & 10.21 & 21.51 & 10.21 \\
0.52 & 1.10 & 2.30 & 4.85 & 10.21 & 21.51
\end{pmatrix}
$$

Correlation Matrix for $\boldsymbol{Y}_i$

$$\widehat{\text{corr}}_{\boldsymbol{y}} = \begin{pmatrix} 1 & & & & & \\ .45 & 1 & & & & \\ .20 & .52 & 1 & & & \\ .08 & .32 & .61 & 1 & & \\ .03 & .25 & .47 & .70 & 1 & \\ .01 & .23 & .42 & .59 & .77 & 1 \end{pmatrix}$$

Correlation Matrix for $\boldsymbol{Y}_i$

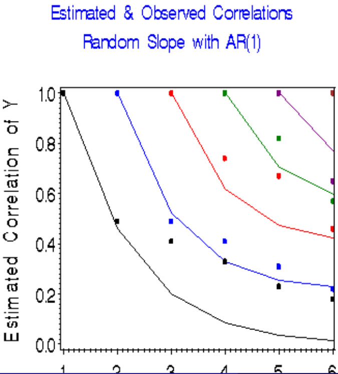
Observed = dots, Estimated = lines

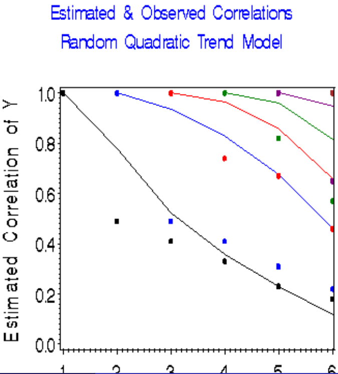


Quadratic Trend or AR(1)?

Observed = dots, Estimated = lines

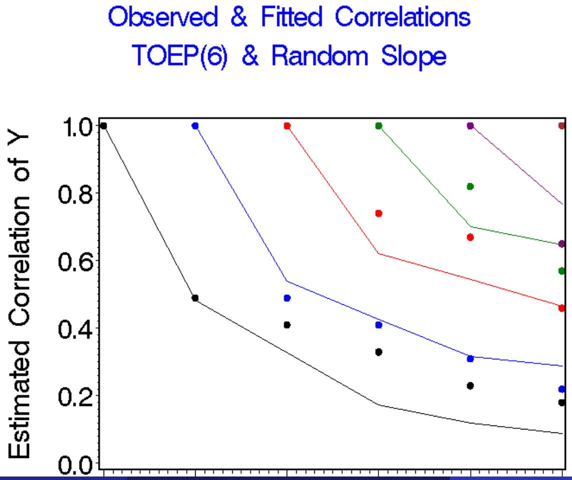
Which is Better?





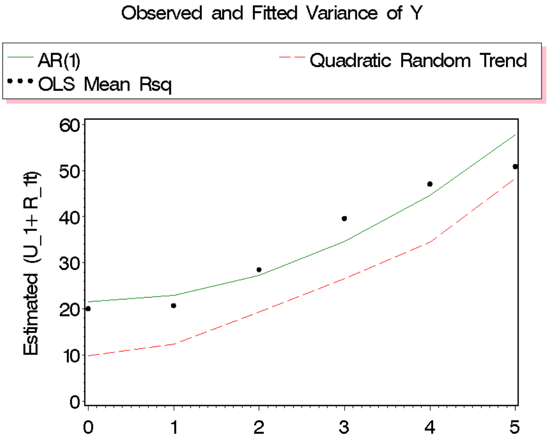
TOEP(6)

Observed = dots, Estimated = lines



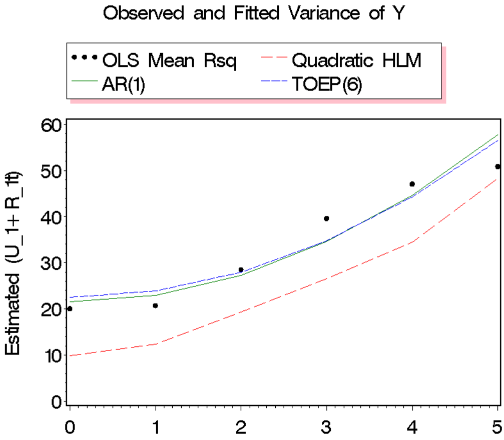
Quadratic Trend or AR(1)?

Observed = dots, Estimated = lines



Quadratic Trend or TOEP(6)

Observed = dots, Estimated = lines



SAS

```
PROC MIXED data=one METHOD=ML noclprint ;
CLASS ID endog week;
MODEL HAMD = time time*time endog / SOLUTION ddfm=sat;
RANDOM intercept time / SUB =ID TYPE=UN;
REPEATED week / subject=ID type=AR(1);
```

- For Moving average,
  `REPEATED week / subject=ID type=TOEP(2)`
- For ARMA,
  `REPEATED week / subject=ID type=ARMA(1,1)`

R

To add AR(1) to HLM use the `nlme` package

```
nlme.ar1 ← lme( hamD ∼ 1 + week + wksq + endog,
      random= ∼ 1 + week + wksq | id,
      correlation=corAR1 (form = ∼ 1 + week |id),
      data=hd )
```

- For ARMA(1,1),
  `correlation=corARMA (form= ∼ 1 + week |id, p=1,q=1)`
- For Moving average, need to do program form of matrix.
- To get correspondences between SAS and R, need to use REML in SAS.
- Implication for likelihood ratio tests, AIC and BIC?

Random Effects vs Serial Correlation

- Models with random effects and unstructured correlation matrices will (always?) fit better than the simpler ones with serial correlation structures.
- Random effect models lead to heteroscedasticity and covariance (correlational) structures we would expect with serial dependence.
- Models with the same fixed effects structures but different covariance structures will
  - Have approximately the same estimates for the fixed effects.
  - The model with the better covariance structure will have smaller S.E.'s for the fixed effects (more efficient).
- Let the data decide!

Random Effects vs Serial Correlation

- Snijders & Bosker:
- Riesby data: found a reasonable model with complex random effects structure and one with simple random effects with more complex serial correlation structure.
- Does theory suggest one or the other?
- There is a wide range of possible combinations of random effects and serial covariance structures. . .

Possible Covariance Structures for $\boldsymbol{Y}_i$

Some examples that we'll look at

- Only random effects (i.e., $\boldsymbol{U}_i$) — We've looked at these most of the semester.
- Only serial correlation (i.e., $\boldsymbol{R}_i$).
- Both random and serial.

  Most of this is/was from Hedeker's web-site but translated into our notation (any mistakes are mine).

Only Random Effects

Random-intercepts model:

$$\boldsymbol{Z}_i' = (1, 1, \ldots, 1) \qquad \boldsymbol{T} = \tau_0^2 \qquad \boldsymbol{R}_i = \sigma^2 \boldsymbol{I}_i$$

Gives

$$\boldsymbol{\Sigma}_{\boldsymbol{y}_i} = \boldsymbol{Z}_i \boldsymbol{T} \boldsymbol{Z}_i' + \sigma^2 \boldsymbol{I}_i = \begin{pmatrix} \tau_0^2 + \sigma^2 & \tau_0^2 & \ldots & \tau_0^2 \\ \tau_0^2 & \tau_0^2 + \sigma^2 & \ldots & \tau_0^2 \\ \vdots & \vdots & \ddots & \vdots \\ \tau_0^2 & \tau_0^2 & \ldots & \tau_0^2 + \sigma^2 \end{pmatrix}$$

Random Intercept & Random Slope

Random linear trend:

$$\boldsymbol{Z}_i' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \boldsymbol{T} = \begin{pmatrix} \tau_0^2 & \tau_{01} \\ \tau_{01} & \tau_1^2 \end{pmatrix} \qquad \boldsymbol{R}_i = \sigma^2 \boldsymbol{I}_i$$

Gives

$$\begin{aligned} \boldsymbol{\Sigma}_{\boldsymbol{y}_i} &= \boldsymbol{Z}_i \boldsymbol{T} \boldsymbol{Z}_i' + \sigma^2 \boldsymbol{I}_i \\ & \begin{pmatrix} \tau_0^2 + \sigma^2 & \tau_0^2 + \tau_{01} & \tau_0^2 + 2\tau_{01} \\ \tau_0^2 + \tau_{01} & \tau_0^2 + 2\tau_{01} + \tau_1^2 + \sigma^2 & \tau_0^2 + 3\tau_{01} + 2\tau_1^2 \\ \tau_0^2 + 2\tau_{01} & \tau_0^2 + 3\tau_{01} + 2\tau_1^2 & \tau_0^2 + 4\tau_{01} + 4\tau_1^2 + \sigma^2 \end{pmatrix} \end{aligned}$$

Random Intercept & Random Slopes

Random quadratic trend:

$$\boldsymbol{R}_i = \sigma^2 \boldsymbol{I}_i \qquad \boldsymbol{Z}_i' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 4 \end{pmatrix} \qquad \boldsymbol{T} = \begin{pmatrix} \tau_0^2 & \tau_{01} & \tau_{02} \\ \tau_{01} & \tau_1^2 & \tau_{12} \\ \tau_{02} & \tau_{12} & \tau_2^2 \end{pmatrix}$$

Gives $\boldsymbol{\Sigma}_{\boldsymbol{y}_i}$

$$\begin{pmatrix} \tau_0^2 + \sigma^2 & \tau_0^2 + \tau_{01} + \tau_{02} & \tau_0^2 + 2\tau_{01} + 4\tau_{02} \\ \tau_0^2 + \tau_{01} + \tau_{02} & \tau_0^2 + 2\tau_{01} + \tau_1^2 + +2\tau_{12}\tau_2^2 + \sigma^2 & \tau_0^2 + 3\tau_{01} + 2\tau_1^2 + 5\tau_{02} + 6\tau_{12} + 4\tau_2^2 \\ \tau_0^2 + 2\tau_{01} + 4\tau_{02} & \tau_0^2 + 3\tau_{01} + 2\tau_1^2 + 5\tau_{02} + 6\tau_{12} + 4\tau_2^2 & \tau_0^2 + 4\tau_{01} + 4\tau_1^2 + 8\tau_{02} + 16\tau_{12} + \sigma^2 \end{pmatrix}$$

Only Model for $\boldsymbol{R}_i$

Compound symmetry: CS

$$\boldsymbol{\Sigma}_{\boldsymbol{y}_i} = \begin{pmatrix} \sigma^2 + \sigma_1^2 & \sigma_1^2 & \sigma_1^2 \\ \sigma_1^2 & \sigma^2 + \sigma_1^2 & \sigma_1^2 \\ \sigma_1^2 & \sigma_1^2 & \sigma^2 + \sigma_1^2 \end{pmatrix}.$$

How else can you get this structure?
Random intercept model, i.e., $\sigma_1^2 = \tau_0^2$.

Only Model for $\boldsymbol{R}_i$

Frist-Order autoregressive: AR(1)

$$\boldsymbol{\Sigma}_{\boldsymbol{y}_i} = \sigma^{2*} \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix}.$$

Only Model for $\boldsymbol{R}_i$ (continued)

Second-Order Toeplitz: TOEP(2)

$$\boldsymbol{\Sigma}_{\boldsymbol{y}_i} = \begin{pmatrix} \sigma_1^2 & \sigma_2^2 & 0 \\ \sigma_2^2 & \sigma_1^2 & \sigma_2^2 \\ 0 & \sigma_2^2 & \sigma_1^2 \end{pmatrix}.$$

General Toeplitz:

$$\boldsymbol{\Sigma}_{\boldsymbol{y}_i} = \begin{pmatrix} \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \\ \sigma_2^2 & \sigma_1^2 & \sigma_2^2 \\ \sigma_3^2 & \sigma_2^2 & \sigma_1^2 \end{pmatrix}.$$

Only Model for $\boldsymbol{R}_i$ (continued)

Unstructured: (UN)

$$\boldsymbol{\Sigma}_{\boldsymbol{y}_i} = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix}.$$

When do we assume this for the covariance matrix of a response variable?
Standard multivariate methods (e.g.,MANOVA)

Implication: A way to deal with missing data.

Random Intercept with Compound Symmetry

$$\boldsymbol{Z}' = (1,1,1) \quad \boldsymbol{T} = \tau_0^2 \quad \text{cov}(\boldsymbol{R}) = CS$$

$$\begin{aligned}
\boldsymbol{\Sigma}_{\boldsymbol{y}_i} &= \boldsymbol{Z}_i \boldsymbol{T} \boldsymbol{Z}_i' + \sigma^2 \boldsymbol{I}_i + \sigma_1^2 \mathbf{1}_i \mathbf{1}_i' \\
&= \begin{pmatrix} \tau_0^2 & \tau_0^2 & \tau_0^2 \\ \tau_0^2 & \tau_0^2 & \tau_0^2 \\ \tau_0^2 & \tau_0^2 & \tau_0^2 \end{pmatrix} + \begin{pmatrix} \sigma^2 + \sigma_1^2 & \sigma_1^2 & \sigma_1^2 \\ \sigma_1^2 & \sigma^2 + \sigma_1^2 & \sigma_1^2 \\ \sigma_1^2 & \sigma_1^2 & \sigma^2 + \sigma_1^2 \end{pmatrix} \\
&= \begin{pmatrix} \sigma^2 + \sigma_1^2 + \tau_0^2 & \sigma_1^2 + \tau_0^2 & \sigma_1^2 + \tau_0^2 \\ \sigma_1^2 + \tau_0^2 & \sigma^2 + \sigma_1^2 + \tau_0^2 & \sigma_1^2 + \tau_0^2 \\ \sigma_1^2 + \tau_0^2 & \sigma_1^2 + \tau_0^2 & \sigma^2 + \sigma_1^2 + \tau_0^2 \end{pmatrix}
\end{aligned}$$

...Just compound symmetry?

Random Intercept with . . .

First Order Autoregressive

$$\boldsymbol{Z}' = (1,1,1) \quad \boldsymbol{T} = \tau_0^2 \quad \text{cov}(\boldsymbol{R}) = AR(1)$$

$$\begin{aligned}
\boldsymbol{\Sigma}_{\boldsymbol{y}_i} &= \begin{pmatrix} \tau_0^2 & \tau_0^2 & \tau_0^2 \\ \tau_0^2 & \tau_0^2 & \tau_0^2 \\ \tau_0^2 & \tau_0^2 & \tau_0^2 \end{pmatrix} + \sigma^{2*} \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^* & \rho & 1 \end{pmatrix} \\
&= \begin{pmatrix} \sigma^{2*} + \tau_0^2 & \rho\sigma^{2*} + \tau_0^2 & \rho^2\sigma^{2*} + \tau_0^2 \\ \rho\sigma^{2*} + \tau_0^2 & \sigma^{2*} + \tau_0^2 & \rho\sigma^{2*} + \tau_0^2 \\ \rho^2\sigma^{2*} + \tau_0^2 & \rho\sigma^{2*} + \tau_0^2 & \sigma^{2*} + \tau_0^2 \end{pmatrix}
\end{aligned}$$

Constant variance, constant (but differing) bands, decreasing covariances.

Random Intercept with ...

Toeplitz Errors

$$\boldsymbol{Z}' = (1,1,1) \quad \boldsymbol{T} = \tau_0^2 \quad \text{cov}(\boldsymbol{R}) = TOEP()$$

$$\begin{aligned}
\boldsymbol{\Sigma}_{\boldsymbol{y}_i} &= \begin{pmatrix} \tau_0^2 & \tau_0^2 & \tau_0^2 \\ \tau_0^2 & \tau_0^2 & \tau_0^2 \\ \tau_0^2 & \tau_0^2 & \tau_0^2 \end{pmatrix} + \begin{pmatrix} \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \\ \sigma_2^2 & \sigma_1^2 & \sigma_2^2 \\ \sigma_3^2 & \sigma_2^2 & \sigma_1^2 \end{pmatrix} \\
&= \begin{pmatrix} \sigma_1^2 + \tau_0^2 & \sigma_2^2 + \tau_0^2 & \sigma_3^2 + \tau_0^2 \\ \sigma_2^2 + \tau_0^2 & \sigma_1^2 + \tau_0^2 & \sigma_2^2 + \tau_0^2 \\ \sigma_3^2 + \tau_0^2 & \sigma_2^2 + \tau_0^2 & \sigma_1^2 + \tau_0^2 \end{pmatrix}
\end{aligned}$$

Constant variance, constant (by differing) bands, decreasing covariances.

Random Linear Trend with . . .

Compound symmetry: $\text{cov}(\boldsymbol{R}_i) = CS$

$$\boldsymbol{Z}_i' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \boldsymbol{T} = \begin{pmatrix} \tau_0^2 & \tau_{01} \\ \tau_{01} & \tau_1^2 \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{\Sigma}_{\boldsymbol{y}_i} \;&=\; \begin{pmatrix} \tau_0^2 & \tau_0^2+\tau_{01} & \tau_0^2+2\tau_{01} \\ \tau_0^2+\tau_{01} & \tau_0^2+2\tau_{01}+\tau_1^2 & \tau_0^2+3\tau_{01}+2\tau_1^2 \\ \tau_0^2+2\tau_{01} & \tau_0^2+3\tau_{01}+2 & \tau_0^2+3\tau_{01}+4\tau_1^2 \end{pmatrix} \\ &\quad + \begin{pmatrix} \sigma^2+\sigma_1^2 & \sigma_1^2 & \sigma_1^2 \\ \sigma_1^2 & \sigma^2+\sigma_1^2 & \sigma_1^2 \\ \sigma_1^2 & \sigma_1^2 & \sigma^2+\sigma_1^2 \end{pmatrix} \end{aligned}$$

Random Linear Trend with . . .

Compound symmetry: $\text{cov}(\boldsymbol{R}_i) = CS$

$$\boldsymbol{Z}_i' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \boldsymbol{T} = \begin{pmatrix} \tau_0^2 & \tau_{01} \\ \tau_{01} & \tau_1^2 \end{pmatrix}$$

$$\boldsymbol{\Sigma}_{\boldsymbol{y}_i} = \begin{pmatrix} \sigma^2 + \sigma_1^2 + \tau_0^2 & \sigma_1^2 + \tau_0^2 + \tau_{01} & \sigma_1^2 + \tau_0^2 + 2\tau_{01} \\ \sigma_1^2 + \tau_0^2 + \tau_{01} & \sigma^2 + \sigma_1^2 + \tau_0^2 + 2\tau_{01} + \tau_1^2 & \sigma_1^2 + \tau_0^2 + 3\tau_{01} + 2\tau_1^2 \\ \sigma_1^2 + \tau_0^2 + 2\tau_{01} & \sigma_1^2 + \tau_0^2 + 3\tau_{01} + 2 & \sigma^2 + \sigma_1^2 + \tau_0^2 + 3\tau_{01} + 4\tau_1^2 \end{pmatrix}$$

Increasing variances, non-constant covariances.

Random Linear Trend with . . .

First Order Autoregressive: $\boldsymbol{R}_i = AR(1)$

$$\boldsymbol{Z}_i' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \boldsymbol{T} = \begin{pmatrix} \tau_0^2 & \tau_{01} \\ \tau_{01} & \tau_1^2 \end{pmatrix}$$

$$\begin{aligned} \boldsymbol{\Sigma}_{\boldsymbol{y}_i}^2 &= \begin{pmatrix} \tau_0^2 & \tau_0^2 + \tau_{01} & \tau_0^2 + 2\tau_{01} \\ \tau_0^2 + \tau_{01} & \tau_0^2 + 2\tau_{01} + \tau_1^2 & \tau_0^2 + 3\tau_{01} + 2\tau_1^2 \\ \tau_0^2 + 2\tau_{01} & \tau_0^2 + 3\tau_{01} + 2 & \tau_0^2 + 3\tau_{01} + 4\tau_1^2 \end{pmatrix} \\ &\quad + \sigma^{2*} \begin{pmatrix} 1 & \rho & \rho^2 \\ \rho & 1 & \rho \\ \rho^2 & \rho & 1 \end{pmatrix} \end{aligned}$$

Random Linear Trend with . . .

First Order Autoregressive: $\boldsymbol{R}_i = AR(1)$

$$\boldsymbol{Z}_i' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \boldsymbol{T} = \begin{pmatrix} \tau_0^2 & \tau_{01} \\ \tau_{01} & \tau_1^2 \end{pmatrix}$$

$$\boldsymbol{\Sigma}_{\boldsymbol{y}_i} = \begin{pmatrix} \sigma^{2*}\tau_0^2 & \rho\sigma^{2*}(\tau_0^2+\tau_{01}) & \rho^2\sigma^{2*}(\tau_0^2+2\tau_{01}) \\ \rho\sigma^{2*}(\tau_0^2+\tau_{01}) & \sigma^{2*}(\tau_0^2+2\tau_{01}+\tau_1^2) & \rho\sigma^{2*}(\tau_0^2+3\tau_{01}+2\tau_1^2) \\ \rho^2\sigma^{2*}(\tau_0^2+2\tau_{01}) & \rho\sigma^{2*}(\tau_0^2+3\tau_{01}+2) & \sigma^{2*}(\tau_0^2+3\tau_{01}+4\tau_1^2) \end{pmatrix}$$

Random Linear Trend with ...

Toeplitz Errors: $\boldsymbol{R}_i = TOEP$

$$\boldsymbol{Z}_i' = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \qquad \boldsymbol{T} = \begin{pmatrix} \tau_0^2 & \tau_{01} \\ \tau_{01} & \tau_1^2 \end{pmatrix}$$

$$\begin{aligned}
\boldsymbol{\Sigma}_{\boldsymbol{y}_i} &= \begin{pmatrix} \tau_0^2 & \tau_0^2 + \tau_{01} & \tau_0^2 + 2\tau_{01} \\ \tau_0^2 + \tau_{01} & \tau_0^2 + 2\tau_{01} + \tau_1^2 & \tau_0^2 + 3\tau_{01} + 2\tau_1^2 \\ \tau_0^2 + 2\tau_{01} & \tau_0^2 + 3\tau_{01} + 2 & \tau_0^2 + 3\tau_{01} + 4\tau_1^2 \end{pmatrix} \\
&\quad + \begin{pmatrix} \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \\ \sigma_2^2 & \sigma_1^2 & \sigma_2^2 \\ \sigma_3^2 & \sigma_2^2 & \sigma_1^2 \end{pmatrix} \\
&= \begin{pmatrix} \sigma_1^2 + \tau_0^2 & \sigma_2^2 + \tau_0^2 + \tau_{01} & \sigma_3^2 + \tau_0^2 + 2\tau_{01} \\ \sigma_2^2 + \tau_0^2 + \tau_{01} & \sigma_1^2 + \tau_0^2 + 2\tau_{01} + \tau_1^2 & \sigma_2^2 + \tau_0^2 + 3\tau_{01} + 2\tau_1^2 \\ \sigma_3^2 + \tau_0^2 + 2\tau_{01} & \sigma_2^2 + \tau_0^2 + 3\tau_{01} + 2 & \sigma_1^2 + \tau_0^2 + 3\tau_{01} + 4\tau_1^2 \end{pmatrix}
\end{aligned}$$

How to Decide?

- Lots of possibilities.
- Tools for selection:
  - Look at variances and covariances:

    Constant or non-constant variance?

    Constant "Bands"?

    Decreasing covariances?
  - LR tests for nested models.
  - Information criteria.
  - What you know about the data & processes.
- Look for a set of "good" ones.