Random Intercept Models

Edps/Psych/Soc 589

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Outline

- A very simple case of a random intercept model.
- When to use this model
- The model
  - Empty model
  - One explanatory variable
  - Multiple explanatory variables
- Next: SAS and R overview followed by computer lab 1.

Reading: Snijders & Bosker, pp 38–56

A Simple Model

Situation:

$j = 1, \ldots, N$ groups
(e.g., schools).

$i = 1, \ldots, n_j$ individuals within the groups
(e.g., students).

Y_{ij} a numerical response variable
(e.g., math scores).

x_{ij} a possible explanatory variable
(e.g., SES).

Simple Level 1 Model

$$Y_{ij} = \beta_{0j} + \beta_1 x_{1ij} + R_{ij}$$

where

- β_1 is fixed in the population.
- $R_{ij} \sim \mathcal{N}(0, \sigma^2)$ and independent.
- The intercept, β_{0j}, depends on the group.

Simple Level 2 Model

The intercept, β_{0j}, can be broken down into two parts:

- An overall or average value of the intercept:

$$\gamma_{00}$$

- A group dependent part of the intercept:

$$U_{0j}$$

- The Level 2 Model is

$$\beta_{0j} = \gamma_{00} + U_{0j}$$

The Linear Mixed Model

Replace β_{0j} in the level 1 model by the level 2 model for β_{0j}:

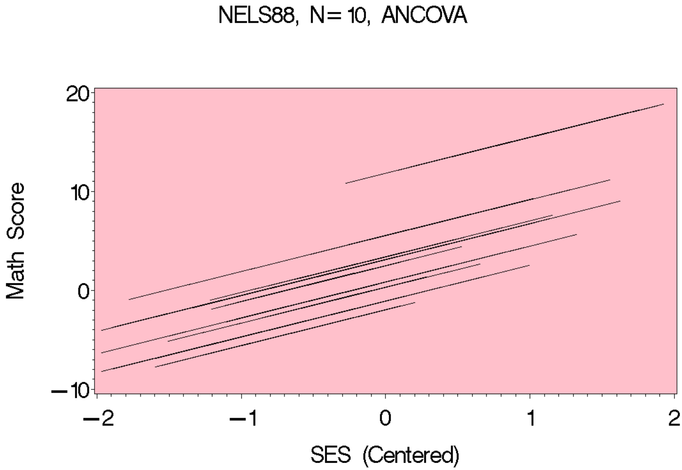
$$\begin{aligned} Y_{ij} &= \gamma_{00} + U_{0j} + \beta_1 x_{1ij} + R_{ij} \\ &= \gamma_{00} + U_{0j} + \gamma_{10} x_{1ij} + R_{ij} \end{aligned}$$

where $\beta_1 = \gamma_{10}$

(to be consistent with later models).

NELS88 Data

What the model might look like applied to the NELS88 data ($N = 10$



Two cases of the model

$$Y_{ij} = \gamma_{00} + \gamma_{10} x_{1ij} + U_{0j} + R_{ij}$$

Case 1: ANCOVA — If the U_{0j}'s are fixed parameters in the population (i.e., The U_{0j}'s are not random but are the group effects).

Case 2: Random intercept model (i.e., HLM) — If the U_{0j}'s are random; that is, groups are a random sample in the population.

Contrasting ANCOVA & HLM

ANCOVA/ANOVA: Groups are unique and interest is only in the groups observed.	Random Coefficients Model: Groups are a sample from a (real or hypothetical) population and
A Group's observations provide information about U_{0j}, so with small n_j, don't get precise estimates of group effects (large standard errors).	Group effects come from the same distribution, data from all groups is used $\Longrightarrow$ can do well with small n_j.
In ANCOVA, U_{0j}'s represent all the differences between groups (no unexplained variability left).	Synonyms: "random effects" & "unexplained variability" i.e., $U_{0j} \sim \mathcal{N}(0, \tau_0^2)$, groups are "exchangeable".
If U_{0j} and R_{ij} are non-normal and/or dependent, random coefficient model may be unreliable.	Can use a different distribution.

Random Intercept Model: no x's

The baseline/empty/null HLM (no explanatory variables).

A regression model where the intercept is a random variable.
Models for Clustered Data
Level 1:

$$Y_{ij} = \beta_{0j} + R_{ij}$$

where

- $R_{ij} \sim \mathcal{N}(0, \sigma^2)$, and independent.
- The intercept, β_{0j}, is a random variable for which we specify a linear regression model (level 2, e.g., groups).

Level 2:

$$\beta_{0j} = \gamma_{00} + U_{0j}$$

where

- γ_{00} is the intercept (fixed).
- $U_{0j} \sim \mathcal{N}(0, \tau_0^2)$ and independent.
- U_{0j} and R_{ij} are independent.

Levels 1 and 2

Estimation of level 1 and level 2 models should be simultaneous.

Substituting our equation for β_{0j} (level 2 model) into the equation for level 1, we get a composite or linear mixed model

$$\begin{aligned} Y_{ij} &= \qquad \beta_{oj} \qquad + R_{ij} \\ &= \underbrace{\gamma_{00}}_{\text{fixed}} + \underbrace{U_{0j} + R_{ij}}_{\text{random}} \end{aligned}$$

and the marginal model is

$$Y_{ij} \sim \mathcal{N}(\gamma_{00}, (\tau_0^2 + \sigma^2))$$

Estimation

$$\begin{aligned} Y_{ij} &= \beta_{oj} + R_{ij} \\ &= \underbrace{\gamma_{00}}_{\text{fixed}} + \underbrace{U_{0j} + R_{ij}}_{\text{random}} \end{aligned}$$

and

$$Y_{ij} \sim \mathcal{N}(\gamma_{00}, (\tau_0^2 + \sigma^2))$$

- This (marginal model) is what is used to fit the model to data and in SAS/MIXED or R, we specify the linear mixed model.
- To use HLM7 software, you specify the component models (i.e., the level 1 model, and the level 2 models) and the program figures out what the marginal model is and fits it to the data.

Example: Null Model

High School and Beyond: A nationally representative sub-sample from the 1982 High School and Beyond Survey. It includes

- Information on $n_+ = 7185$ students.
- Students are nested within $N = 160$ schools:
  90 public and 70 Catholic.
- Samples sizes averaged about $n_j \sim 45$ per school.

Example: Level 1 Level 2 variables

Level 1 Variables: Student level variables

Outcome variable: Math achievement.

Explanatory:

- Student socioeconomic status (SES) — composite of parental education, occupation and income. This have been standardized such that average equals 0.
- Student ethnicity (1=minority, 0=not).
- Gender (1=female, 0=male).

Level 2 Variables: School level variables

- Sector — whether the school is Catholic $(= 1)$ or public $(= 0)$.
- Mean SES — average SES of students w/in a school.
- School enrollment.
- Proportion of students in academic track.
- Disciplinary climate.
- Whether the percentage minority is $\leq 40\%$ or $> 40\%$.

Example: the null HLM

Y_{ij} = math achievement.

Groups are schools: $j = 1, \ldots, 160$.

Diagram representing the situation:

. . . .

math

The Hierarchical Null Model

Level 1:

$$\begin{aligned} Y_{ij} &= \beta_{0j} + R_{ij} \\ (\text{math})_{ij} &= \beta_{0j} + R_{ij} \end{aligned}$$

where $R_{ij} \sim \mathcal{N}(0, \sigma^2)$ and independent.

Level 2:

$$\beta_{0j} = \gamma_{00} + U_{0j}$$

where $U_{0j} \sim \mathcal{N}(0, \tau_0^2)$ and independent.

…and R_{ij} and U_{0j} are independent.

The Marginal Model

The Linear Mixed model:

$$(\text{math})_{ij} = Y_{ij} = \gamma_{00} + U_{0j} + R_{ij}$$

where U_{0j} and R_{ij} are independent and

The Marginal Model:

$$Y_{ij} \sim \mathcal{N}(\gamma_{00}, (\tau_0^2 + \sigma^2)).$$

Implications for data.

Results 1

Some descriptive statistics:

The MEANS Procedure

Variable	Label	Mean	StdDev
mathach	Mathematics achievement	12.748	6.878
ses	standardized student ses	0.000	0.779

Or in R, `means(mathach), sd(mathach)`, where `mathach` is the name of math scores in hsb dataframe.

Results 2

The Mixed Procedure

Model Information

Data Set	SASDATA.HSBALL
Dependent Variable	mathach
Covariance Structure	Variance Components
Subject Effect	id
Estimation Method	ML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Containment

Results 3

Dimensions

Covariance Parameters	2
Columns in X	1
Columns in Z Per Subject	1
Subjects	160
Max Obs Per Subject	67
Observations Used	7185
Observations Not Used	0
Total Observations	7185

Results 4

Iteration History

Iteration	Evaluations	-2 Log Like	Criterion
0	1	48099.73204627	
1	2	47115.82988208	0.00000114
2	1	47115.81024259	0.00000000

Convergence criteria met.

Covariance Parameter Estimates

Cov Parm	Subject	Estimate	Standard Error	Z Value	Pr $Z > \|t\|$
Intercept	id	8.5490	1.0676	8.01	$< .0001$
Residual		39.1488	0.6607	59.26	$< .0001$

Results 5

Fit Statistics	
-2 Log Likelihood	47115.8
AIC (smaller is better)	47121.8
AICC (smaller is better)	47121.8
BIC (smaller is better)	47131.0

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr $>|t|$
Intercept	12.6371	0.2436	159	51.88	$< .0001$

Summary of Results

Null Model

Parameter	Value	SE	Parameter	Value	SE
γ_{00}	12.64	.24	τ_0^2	8.55	1.07
			σ^2	39.15	.66

Estimate of the intra-class correlation,

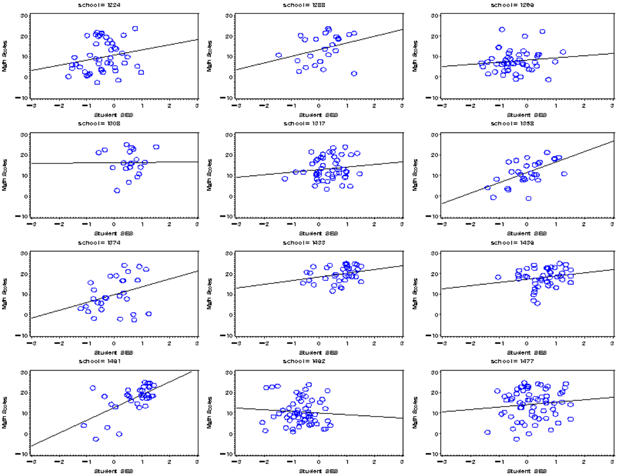
$$\hat{\rho}_I = \frac{\hat{\tau}^2}{\hat{\tau}^2 + \hat{\sigma}^2} = \frac{8.55}{8.55 + 39.15} = \frac{8.55}{47.70} = .18$$

and $\widehat{(\text{math})}_{ij} = 12.64$, the overall (total sample) mean,

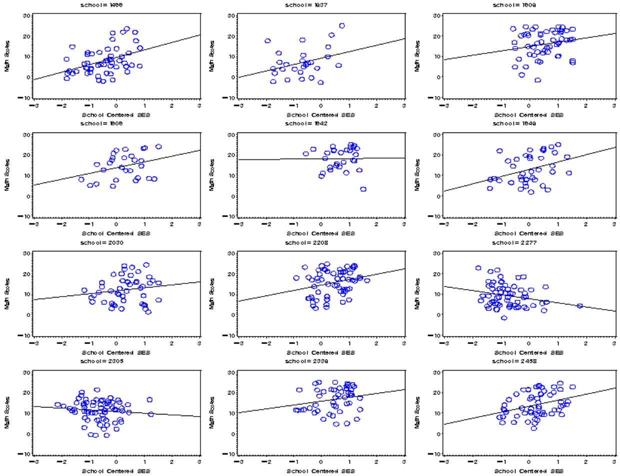
The overall variance of math scores is $s^2 = (8.55 + 39.15) = 47.70$.

Note $s = 6.873$ and $s^2 = 47.301$— see descriptive statistics, p 18.

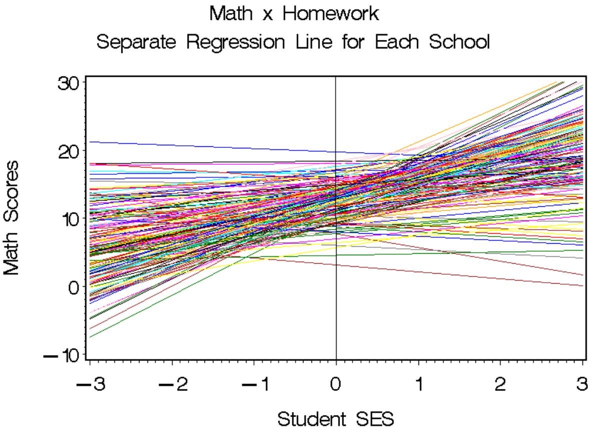
Add SES as Level 1 Explanatory?



Some More Schools



Overlay Regressions for all Schools



Random Intercept: One x

SES to help explain some of the variability of Y_{ij}.

The Hierarchical Model:

Level 1:

$$Y_{ij} = \beta_{0j} + \beta_{1j} x_{ij} + R_{ij}$$

where $R_{ij} \sim \mathcal{N}(0, \sigma^2)$ and independent.

Level 2:

$$\begin{aligned} \beta_{0j} &= \gamma_{00} + U_{0j} \\ \beta_{1j} &= \gamma_{10} \end{aligned}$$

where $U_{0j} \sim \mathcal{N}(0, \tau_0^2)$ and independent (and independent w/rt R_{ij}).

Random Intercept: 1 x

The appropriate diagram to represent this model?

$$\overset{\cdots\cdots\cdots\cdots\cdots\cdots}{\text{SES}_{ij} \longrightarrow \quad \text{math}_{ij}}$$

or more generally

$$\overset{\cdots\cdots\cdots\cdots}{x_{ij} \longrightarrow y_{ij}}$$

Random Intercept: 1 x

The Linear Mixed Model:

$$Y_{ij} = \gamma_{00} + \gamma_{10} x_{ij} + U_{0j} + R_{ij}$$

where

- γ_{00} is the intercept for the average group.
- γ_{10} is regression coefficient for x_{ij} (fixed).
- $U_{0j} \sim \mathcal{N}(0, \tau_0^2)$ and independent.
- $R_{ij} \sim \mathcal{N}(0, \sigma^2)$ and independent.
- U_{0j} and R_{ij} are independent.

Notes:

- $\text{var}(Y_{ij}|x_{ij}) = \text{var}(U_{0j}) + \text{var}(R_{ij}) = \tau_0^2 + \sigma^2$
- $\text{cov}(Y_{ij}, Y_{i'j}|x_{ij}, x_{i'j}) = \tau_0^2$
- Residual intra-class correlation,

$$\rho_I(Y|X = x) = \frac{\tau_0^2}{\tau_0^2 + \sigma^2}$$

- If $\tau_0^2 = 0$, then simply use ordinary linear regression (OLS).

Marginal vs Hierarchical Model

The Marginal Model:

$$Y_{ij} \sim \mathcal{N}(\gamma_{00} + \gamma_{10} x_{ij}, (\tau_o^2 + \sigma^2))$$

- The hierarchical model implies the marginal model.
- The marginal model does not imply the hierarchical model.
- There are no random effects in the marginal model.
- The HLM is more restrictive than the marginal.

Linear Mixed Model in Matrix Notation

The linear mixed model,

$$Y_{ij} = \gamma_{00} + \gamma_{10} x_{ij} + U_{0j} + R_{ij},$$

In matrix notation, for marco unit (group) j,

$$\begin{pmatrix} Y_{1j} \\ Y_{2j} \\ \vdots \\ Y_{n_j j} \end{pmatrix} = \begin{pmatrix} 1 & x_{1j} \\ 1 & x_{2j} \\ \vdots & \vdots \\ 1 & x_{n_j j} \end{pmatrix} \begin{pmatrix} \gamma_{00} \\ \gamma_{10} \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} U_{0j} \end{pmatrix} + \begin{pmatrix} R_{1j} \\ R_{2j} \\ \vdots \\ R_{n_j j} \end{pmatrix}$$

$$\boldsymbol{Y}_j = \boldsymbol{X}_j \boldsymbol{\Gamma} + \boldsymbol{Z}_j \boldsymbol{U}_j + \boldsymbol{R}_j$$

Marginal Model in Matrix Notation

And the marginal model is

$$\boldsymbol{Y}_j \sim \mathcal{N}(\boldsymbol{X}_j\boldsymbol{\Gamma}, (\boldsymbol{Z}_j\boldsymbol{T}\boldsymbol{Z}_j' + \sigma^2\boldsymbol{I})).$$

Note that $\boldsymbol{Z}_j\boldsymbol{T}\boldsymbol{Z}_j'$ is just $(n_j \times n_j)$

$$\mathbf{1}\tau^2\mathbf{1}' = \begin{pmatrix} \tau^2 & \tau^2 & \ldots & \tau^2 \\ \tau^2 & \tau^2 & \ldots & \tau^2 \\ \vdots & \vdots & \ddots & \vdots \\ \tau^2 & \tau^2 & \ldots & \tau^2 \end{pmatrix}$$

Example: HSB

$x_{ij} = \text{SES}_{ij}$ (student/micro level).

Some descriptive statistics:

The MEANS Procedure

Variable	N	Mean	Std Dev
Mathematics achievement	7185	2.75	6.88
Standardized student ses	7185	0.00	0.78

Edited output from SAS/MIXED

The Mixed Procedure

Model Information

Data Set	SASDATA.HSBALL
Dependent Variable	mathach
Covariance Structure	Unstructured
Subject Effect	id
Estimation Method	ML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Containment

Example: HSB

Dimensions

Covariance Parameters	2	
Columns in X	2	– Design for fixed
Columns in Z Per Subject	14	– Design for random
Subjects	160	– N
Max Obs Per Subject	67	– largest n_j
Observations Used	7185	– n_+
Observations Not Used	0	
Total Observations	7185	

Convergence criteria met.

Proc MIXED Output

Covariance Parameter Estimates

Cov Parm	Subject	Estimate	Standard Error	Z Value	Pr Z
UN(1,1)	id	4.7268	0.6483	7.29	$< .0001$
Residual		37.0301	0.6253	59.22	$< .0001$

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	$Pr > \|t\|$
Intercept	12.6576	0.1873	159	67.58	$< .0001$
ses	2.3916	0.1057	7024	22.63	$< .0001$

Summary & Comparison

	Null Model		Add SES	
	Value	SE	Value	SE
Fixed effects				
γ_{00}	12.64	.24	12.66	.19
γ_{10}	—	—	2.39	.11
Random effects				
τ_0^2	8.55	1.07	4.73	.65
σ^2	39.15	.66	37.03	.63

Residual intra-class correlation,

$$\hat{\rho}_I(\text{math}|\text{SES}) = \frac{4.73}{4.73 + 37.03} = \frac{4.73}{41.76} = .11$$

Comparison

- Drop in between groups variance estimate $\hat{\tau}_0^2$:

$$\frac{4.73}{8.55} = .55 \quad \text{or} \quad (1 - .56)100 = 45\% \text{ descrease}$$

- Drop in within groups variance estimate $\hat{\sigma}^2$:

$$\frac{37.03}{39.15} = .95 \quad \text{or} \quad (1 - .95)100 = 5\% \text{ descrease}$$

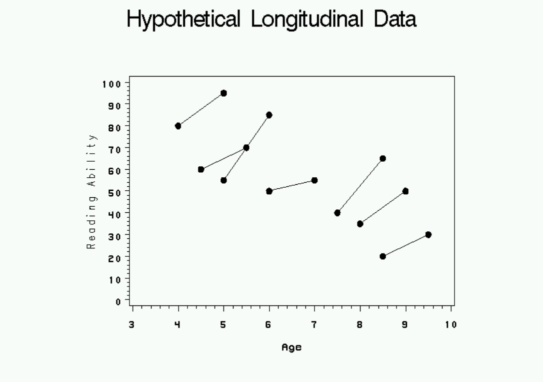
Interpretation somewhat problematic because SES helps to explain both the between and within groups variance of Y_{ij}:

$$\text{SES}_{ij} = \overline{\text{SES}}_j + (\text{SES}_{ij} - \overline{\text{SES}}_j)$$

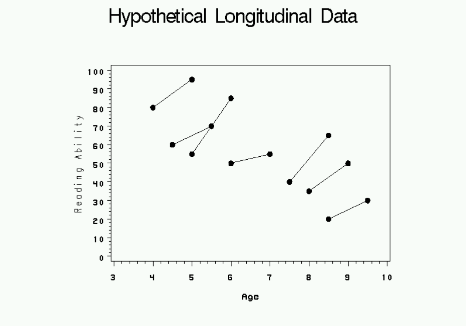
Problematic Interpretation

. . . Between versus Within Group Regressions.
Different processes at work at different levels.

An example:



Problematic Interpretation (continued)

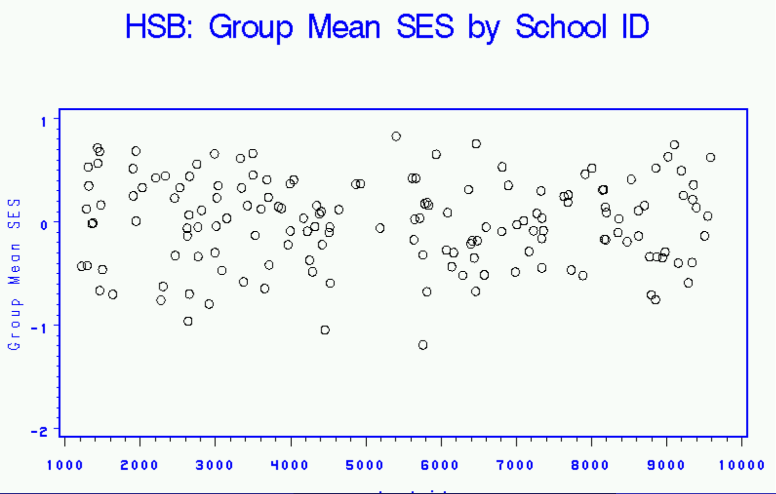


Within individual: Child's reading ability improves with instruction.

Between individuals: Children get tutoring/private instruction for different reasons.

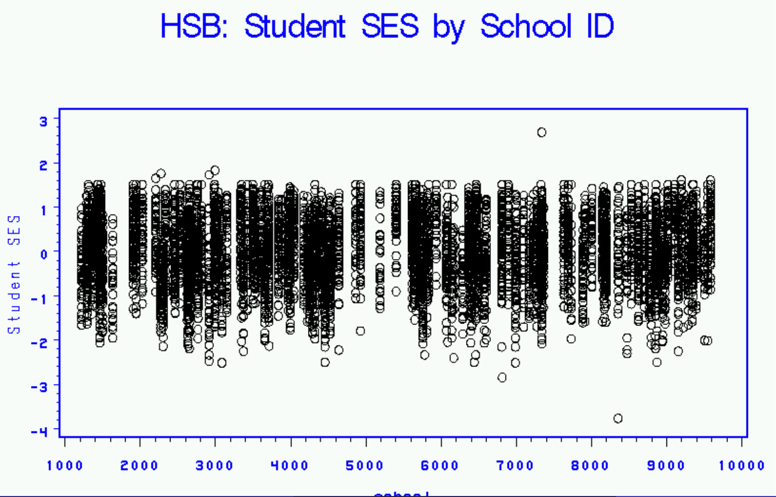
Returning to the HSB Example

Groups differ with respect to mean SES (i.e., $\overline{\text{SES}}_j$).



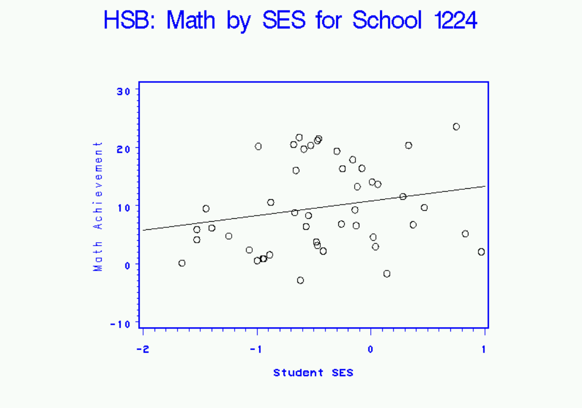
Returning to the HSB Example

Individual students within schools differ with respect to SES.



Returning to the HSB Example

Or for just one school, we have within school variability,



Line is simple linear regression line.

Solution to Problem

Regarding different mean levels of SES between and within schools:

- "Group mean centered" variable, e.g.,

$$x_{ij} = (\text{SES}_{ij} - \overline{\text{SES}}_j)$$

  to model within group variability of Y_{ij} w/rt SES.

- Group mean as a level 2 (school) variable

$$z_j = \overline{\text{SES}}_j$$

Solution to Problem (continued)

Recall that overall mean SES $= 0$.

First we'll just have a level 1 (student) variable for SES:

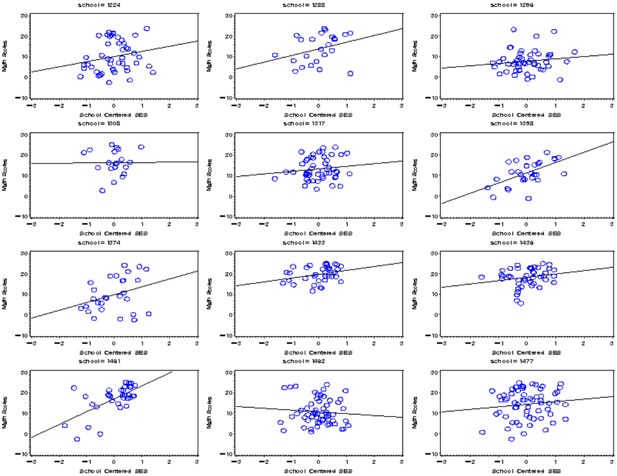
$$x_{ij} = (\mathsf{SES}_{ij} - \overline{\mathsf{SES}}_j)$$

A diagram for this model,

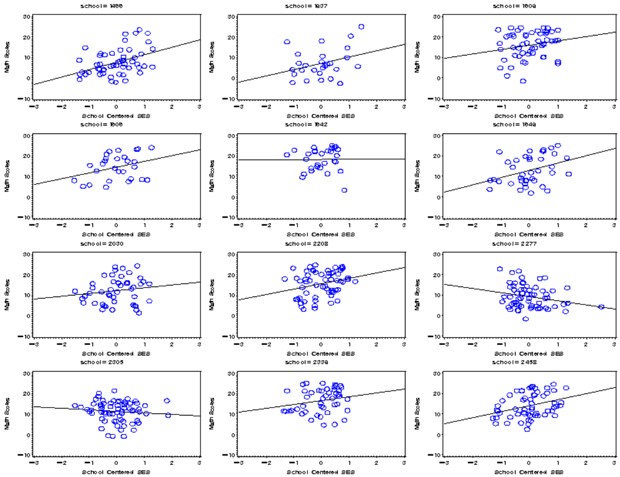
. . . .

$$(\mathsf{SES}_{ij} - \overline{SES}_j) \quad \rightarrow \mathsf{math}_{ij}$$

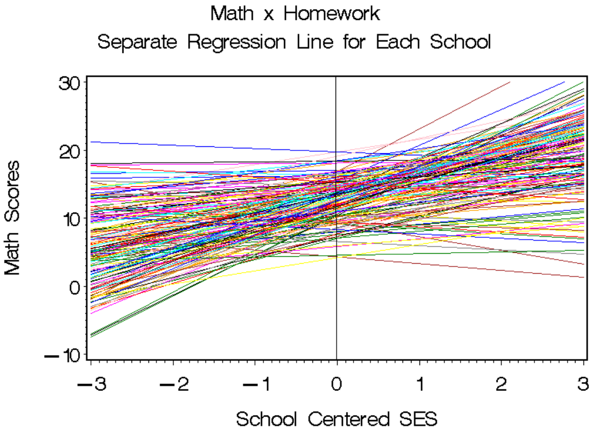
Add Centered SES at Level 1?



Some More Schools



Overlay Regressions for all Schools



Hierarchical Model with Centered SES

Level 1:

$$\begin{aligned} Y_{ij} &= \beta_{0j} + \beta_{1j}x_{ij} + R_{ij} \\ \text{math}_{ij} &= \beta_{0j} + \beta_{1j}(\text{SES}_{ij} - \overline{\text{SES}}_j) + R_{ij} \end{aligned}$$

where $R_{ij} \sim \mathcal{N}(0, \sigma^2)$ and independent.

Level 2:

$$\begin{aligned} \beta_{0j} &= \gamma_{00} + U_{0j} \\ \beta_{1j} &= \gamma_{10} \end{aligned}$$

where $U_{0j} \sim \mathcal{N}(0, \tau_0^2)$ and independent.

Linear Mixed & Marginal Models

The corresponding linear mixed model:

$$\text{math}_{ij} = \gamma_{00} + \gamma_{10}(\text{SES}_{ij} - \overline{\text{SES}}_j) + U_{0j} + R_{ij}$$

and Marginal Model:

$$\text{math}_{ij} \sim \mathcal{N}(\gamma_{00} + \gamma_{10}(\text{SES}_{ij} - \overline{\text{SES}}_j), (\tau_0^2 + \sigma^2)).$$

Edited SAS/MIXED Output

Convergence criteria met.

Covariance Parameter Estimates

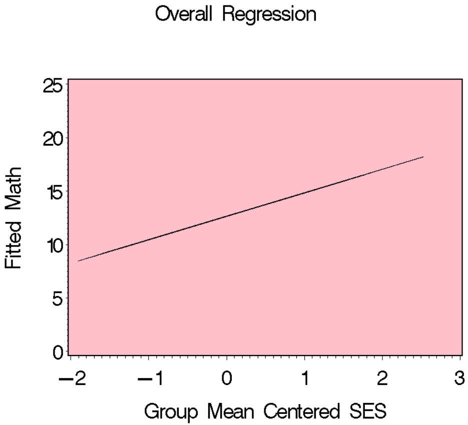
Cov Parm	Subject	Estimate	Standard Error	Z Value	Pr Z
UN(1,1)	id	8.6071	1.0682	8.06	$< .0001$
Residual		37.0056	0.6245	59.26	$< .0001$

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr $> \|t\|$
Intercept	12.6494	0.2437	159	51.92	$< .0001$
cses	2.1912	0.1086	7024	20.17	$< .0001$

Estimated Overall Regression Line

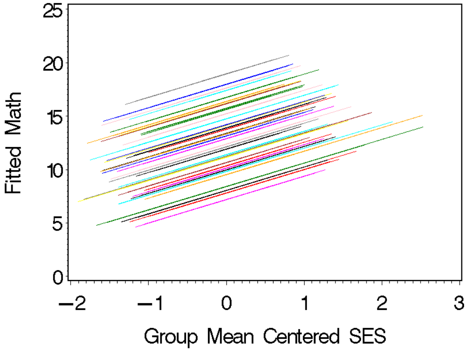
$$\widehat{\text{math}}_{ij} = 12.6494 + 2.1912(\text{SES}_{ij} - \overline{\text{SES}}_j)$$



Sub-Sample of Groups' Lines

$$\widehat{\text{math}}_{ij} = 12.6494 + 2.1912(\text{SES}_{ij} - \overline{\text{SES}}_j) + \hat{U}_j$$

and Some of the 160 Schools



Model Summary & Comparison

	Null Model		Add SES		Add cSES	
	Value	SE	Value	SE	Value	SE
Fixed effects						
γ_{00}	12.64	.24	12.66	.19	12.65	.24
γ_{10}	—	—	2.39	.11	2.19	.11
Random effects						
τ_0^2	8.55	1.07	4.73	.65	8.61	1.07
σ^2	39.15	.66	37.03	.63	37.01	.63

Conclusion

Residual intra-class correlation,

$$\hat{\rho}_I(\text{math}|\text{SES}) = \frac{8.61}{8.61 + 37.01} = \frac{8.61}{45.62} = .19$$

It appears that SES account for some variability in math achievement, Y_{ij}, within schools

$$\left(1 - \frac{37.01}{39.15}\right) \times 100\% = (1 - .945)100\% = 5.5\%$$

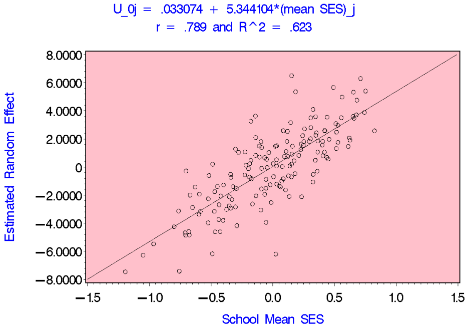
Does $z_j = \overline{SES}_j$ help model between group variability?

Between School Variability & SES

Using $\overline{\text{SES}}_j$ as predictor for β_{oj}.

The $\hat{U}_{0j}$'s plotted below are from

$$Y_{ij} = \underbrace{(\gamma_{00} + U_{0j})} + \gamma_{10}(\text{SES}_{ij} - \overline{\text{SES}}_j) + R_{ij}$$



HLM with x and z

Level 1:

$$\text{math}_{ij} = \beta_{0j} + \beta_{1j}(\text{SES}_{ij} - \overline{\text{SES}}_j) + R_{ij}$$

where $R_{ij} \sim \mathcal{N}(0, \sigma^2)$.

Level 2:

$$\begin{aligned} \beta_{0j} &= \gamma_{00} + \gamma_{01}(\overline{\text{SES}}_j) + U_{0j} \\ \beta_{1j} &= \gamma_{10} \end{aligned}$$

where $U_{0j} \sim \mathcal{N}(0, \tau_0^2)$ and independent.

Linear Mixed Model

$$\text{math}_{ij} = \gamma_{00} + \gamma_{10}(\text{SES}_{ij} - \overline{\text{SES}}_j) + \gamma_{01}(\overline{\text{SES}}_j) + U_{0j} + R_{ij}$$

Guessing What Will Happen

$$\text{math}_{ij} = 12.6494 + 2.1912(\text{SES}_{ij} - \overline{\text{SES}}_j) + U_{0j} + R_{ij}$$

Put in regression of $\hat{U}_{0j}$ on $\overline{\text{SES}}_j$ into above:

$$\begin{aligned} Y_{ij} &= 12.6494 + 2.1912(\text{SES}_{ij} - \overline{\text{SES}}_j) + (.0331 + 5.344(\overline{\text{SES}}_j) + U^*_{0j}) + R_{ij} \\ &= 12.6825 + 2.1912(\text{SES}_{ij} - \overline{\text{SES}}_j) + 5.344(\overline{\text{SES}}_j) + U^*_{0j} + R_{ij} \end{aligned}$$

How about a new τ_0^*?

The R^2 between $\hat{U}_{0j}$ & $\overline{\text{SES}}_j$ equaled $.623$ and previously estimated $\hat{\tau}_0 = 8.6071$.

$$\begin{aligned} \frac{\hat{\tau}_0 - \hat{\tau}_0^*}{\hat{\tau}_0} &\approx R^2 \\ \frac{8.6071 - \hat{\tau}_0^*}{8.6071} &\approx .6233 \longrightarrow \hat{\tau}_0^* \approx 3.24 \end{aligned}$$

Theory and Proposition

In terms of a diagram, our proposition (theory),

$$\overline{\text{SES}}_j$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\searrow\cdots\cdots$$

$$(\text{SES}_{ij} - \overline{\text{SES}}_j) \quad \rightarrow \text{math}_{ij}$$

Micro-micro: Higher a student's SES relative to the school mean, the higher his/her math score.

Macro-micro: Higher average SES of school, higher a student's math score.

Edited SAS/MIXED Output

Dimensions

Covariance Parameters	2	
Columns in X	3	*** this changed
Columns in Z Per Subject	1	
Subjects	160	
Observations Used	7185	
Observations Not Used	0	
Total Observations	7185	
⋮	⋮	

Convergence criteria met.

Edited SAS/MIXED Output

Covariance Parameter Estimates

Cov Parm	Sub	Estimate	Standard Error	Z Value	Pr Z
UN(1,1)	id	2.6427	0.3964	6.67	$< .0001$
Residual		37.0150	0.6247	59.26	$< .0001$

Solution for Fixed Effects

Effect	Estimate	Standard Error	DF	t Value	Pr> $\|t\|$
Intercept	12.6616	0.1483	158	85.37	$< .0001$
cses	2.1912	0.1087	7024	20.17	$< .0001$
meanses	5.8656	0.3591	7024	16.33	$< .0001$

Summary Models Fit to HSB

Null, empty, baseline,

$$\text{math}_{ij} = \gamma_{00} + U_{0j} + R_{ij}$$

1 uncentered variable,

$$\text{math}_{ij} = \gamma_{00} + \gamma_{10}(\text{SES})_{ij} + U_{0j} + R_{ij}$$

Group centered,

$$\text{math}_{ij} = \gamma_{00} + \gamma_{10}(\text{SES}_{ij} - \overline{\text{SES}}_j) + U_{0j} + R_{ij}$$

Group centered and mean,

$$\text{math}_{ij} = \gamma_{00} + \gamma_{10}(\text{SES}_{ij} - \overline{\text{SES}}_j) + \gamma_{01}(\overline{\text{SES}}_j) + U_{0j} + R_{ij}$$

Summary of Parameter Estimates

	Null Model		Add $(\text{SES})_{ij}$		Add cSES_{ij}		Add cSES_{ij} & $(\overline{\text{SES}})_j$	
	Value	SE	Value	SE	Value	SE	Value	SE
Fixed effects								
γ_{00}	12.64	.24	12.66	.19	12.65	.24	12.66	.15
γ_{10}	—	—	2.39	.11	2.19	.11	2.19	.11
γ_{01}	—	—	—	—	—	—	5.87	.36
Random effects								
τ_0^2 (between)	8.55	1.07	4.73	.65	8.61	1.07	2.64	.40
σ^2 (within)	39.15	.66	37.03	.63	37.01	.63	37.02	.62

Note:

$$\text{cSES}_{ij} = (\text{SES})_{ij} - (\overline{\text{SES}})_j$$

$$\overline{\text{SES}}_j = (1/n_j)\sum_{i=1}^{n_j} (\text{SES})_{ij}$$

Two more models

- Add all the student level variables that we have to see if we can account for more of the within groups residual variance.

  i.e., Add more x_{ij}'s to the model.

  - $\text{SES}_{ij} - \overline{\text{SES}}_j$
  - female ($= 1$ if female, $= 0$ if male)
  - minority ($= 1$ if minority, $= 0$ if not)

- Add all the school level variables to see if we can account for more between groups residual variance.

  i.e., Add more z_j's to the model.

  - himinty $= 40\%$ minority, $0 =< 40\%$ minority
  - pracad $=$ Proportion of students in academic track
  - disclim $=$ Disciplinary climate
  - sector $= 1 =$ Catholic, $0 =$ public
  - size $=$school enrollment

HLM with Lots of Micro Variables

$$\textit{Level1}: \quad (\text{math})_{ij} = \beta_{0j} + \beta_{1j}(\text{SES}_{ij} - \overline{\text{SES}}_j) + \beta_{2j}(\text{female})_{ij} + \beta_{3j}(\text{minority})_{ij} + R_{ij}$$

where $R_{ij} \sim \mathcal{N}(0, \sigma^2)$ and independent.

$$\begin{aligned} \textit{Level2}: \quad \beta_{0j} &= \gamma_{00} + \gamma_{01}(\overline{\text{SES}})_j + U_{0j} \\ \beta_{1j} &= \gamma_{10} \\ \beta_{2j} &= \gamma_{20} \\ \beta_{3j} &= \gamma_{30} \end{aligned}$$

where $U_{0j} \sim \mathcal{N}(0, \tau_0^2)$ and independent.
R_{ij} and U_j are independent.

Homework: What's the linear mixed model? What's the marginal model?

Edited SAS/MIXED output

The Mixed Procedure

Model Information

Data Set	WORK.HSBCENT
Dependent Variable	mathach
Covariance Structure	Unstructured
Subject Effect	id
Estimation Method	ML
Residual Variance Method	Profile
Fixed Effects SE Method	Model-Based
Degrees of Freedom Method	Containment

Edited SAS/MIXED output

Dimensions

Covariance Parameters	2
Columns in X	7
Columns in Z Per Subject	1
Subjects	160
Max Obs Per Subject	67
Observations Used	7185
Observations Not Used	0
Total Observations	7185

Edited SAS/MIXED output

Convergence criteria met.

Covariance Parameter Estimates

Cov Parm	Subject	Estimate	Standard Error	Z Value	Pr Z
UN(1,1)	id	2.3921	0.3670	6.52	$< .0001$
Residual		35.8871	0.6057	59.25	$< .0001$

Edited SAS/MIXED output

Solution for Fixed Effects

Effect	Student ethnicity (1=minority, 0=not)	student gender (1=female, 0=male)	Estimate	Standard Error
Intercept			10.1017	0.2197
cses			1.9265	0.1084
minority	0		2.7280	0.2032
minority	1		0	.
female		0	1.2186	0.1607
female		1	0	.
meanses			4.8086	0.3521

Summary & Comparison

	Null Model		Add cSES$_{ij}$		Add cSES$_{ij}$ & $(\overline{\text{SES}})_j$		Add all mirco level	
	Value	SE	Value	SE	Value	SE	Value	SE
Fixed effects								
intercept	12.64	.24	12.65	.24	12.66	.15	10.10	.22
centered SES	—	—	2.19	.11	2.19	.11	1.93	.11
male	—	—	—	—	—	—	1.22	.16
not minority	—	—	—	—	—	—	2.73	.20
mean SES	—	—	—	—	5.87	.36	4.81	.35
Random effects								
τ_0^2 (between)	8.55	1.07	8.61	1.07	2.64	.40	2.39	.37
σ^2 (within)	39.15	.66	37.01	.63	37.02	.62	35.89	.61

Note:
$\text{cSES}_{ij} = (\text{SES})_{ij} - (\overline{\text{SES}})_j$
$\overline{\text{SES}}_j = (1/n_j)\sum_{i=1}^{n_j}(\text{SES})_{ij}$

Lots of Macro-level Variables

Add all of the school (macro-level) variables:

- himinty $= 40\%$ minority, $0 =< 40\%$ minority
- pracad $=$ Proportion of students in academic track
- disclim $=$ Disciplinary climate
- sector $= 1 =$ Catholic, $0 =$ public
- size $=$school enrollment

HLM for Lots of Macro-level

Level 1:

$$\begin{aligned}(\text{math})_{ij} &= \beta_{0j} + \beta_{1j}(\text{SES}_{ij} - \overline{\text{SES}}_j) + \beta_{2j}(\text{female})_{ij} \\ &\quad + \beta_{3j}(\text{minority})_{ij} + R_{ij}\end{aligned}$$

where $R_{ij} \sim \mathcal{N}(0, \sigma^2)$ and independent.

Level 2:

$$\begin{aligned}\beta_{0j} &= \gamma_{00} + \gamma_{01}(\overline{\text{SES}})_j + \gamma_{02}(\text{himinty})_j + \gamma_{03}(\text{pracad})_j \\ &\quad + \gamma_{04}(\text{disclim})_j + \gamma_{05}(\text{sector})_j + \gamma_{06}(\text{size})_j + U_{0j} \\ \beta_{1j} &= \gamma_{10} \\ \beta_{2j} &= \gamma_{20} \\ \beta_{3j} &= \gamma_{30}\end{aligned}$$

where $U_{0j} \sim \mathcal{N}(0, \tau_0^2)$ and independent,
and R_{ij} and U_j are independent.

Linear Mixed w/ Lots of Macro-level

$$
\begin{aligned}
(\text{math})_{ij} &= \gamma_{00} + \gamma_{10}(\text{SES}_{ij} - \overline{\text{SES}}_j) + \gamma_{20}(\text{female})_{ij} \\
&\quad + \gamma_{30}(\text{minority})_{ij} \\
&\quad + \gamma_{01}(\overline{\text{SES}})_j + \gamma_{02}(\text{himinty})_j + \gamma_{03}(\text{pracad})_j \\
&\quad + \gamma_{04}(\text{disclim})_j + \gamma_{05}(\text{sector})_j + \gamma_{06}(\text{size})_j \\
&\quad + U_{0j} + R_{ij}
\end{aligned}
$$

What's the marginal model?

Edited SAS/MIXED output

The Mixed Procedure

Dimensions

Covariance Parameters	2
Columns in X	13
Columns in Z Per Subject	1
Subjects	160
Max Obs Per Subject	67
Observations Used	7185
Observations Not Used	0
Total Observations	7185

Edited SAS/MIXED output

Convergence criteria met.

Covariance Parameter Estimates

Cov Parm	Subject	Estimate	Standard Error	Z Value	Pr Z
UN(1,1)	id	1.2923	0.2402	5.38	$< .0001$
Residual		35.8805	0.6055	59.26	$< .0001$

Edited SAS/MIXED output: Fixed Effects

Effect	Student ethnicity	student gender	Sector	Est.	Std Error
Intercept				7.8981	0.6060
cses				1.9036	0.1085
minority	not			2.9894	0.2111
minority	minority			0	.
female		male		1.2609	0.1580
female		female		0	.
meanses				3.0820	0.4219
himinty				0.2437	0.3182
pracad				3.0027	0.7992
disclim				−0.3868	0.1795
sector			public	−0.8392	0.3863
sector			Catholic	0	.
size				0.000743	0.00021

Edited SAS/MIXED output

Type 3 Tests of Fixed Effects

Effect	Num DF	Den DF	F Value	Pr $> F$
cses	1	7022	307.65	$< .0001$
minority	1	7022	200.45	$< .0001$
female	1	7022	63.66	$< .0001$
meanses	1	7022	53.36	$< .0001$
himinty	1	7022	0.59	0.4438
pracad	1	7022	14.12	0.0002
disclim	1	7022	4.64	0.0312
sector	1	7022	4.72	0.0299
size	1	7022	12.07	0.0005

Summary & Comparison: Fixed Effects

	Null Model		Add cSES_{ij} & $\overline{(\text{SES})}_j$		Add all mirco level		All micro and macro	
	Value	SE	Value	SE	Value	SE	Value	SE
intercept	12.64	.24	12.66	.15	10.10	.22	7.90	.61
centered SES	.	.	2.19	.11	1.93	.11	1.90	.11
male	.	.	.	.	1.22	.16	1.26	.16
not minority	.	.	.	.	2.73	.20	2.99	.21
mean SES	.	.	5.87	.36	4.81	.35	3.08	.42
himinty	.	.	.	.	.	.	.24	.32
pracd	.	.	.	.	.	.	3.00	.80
disclim	.	.	.	.	.	.	$-.39$	.18
sector	.	.	.	.	.	.	-1.84	.39
size	.	.	.	.	.	.	.0007	.0002

Note: $\text{cSES}\ _{ij} = (\text{SES})_{ij} - \overline{(\text{SES})}_j$

$\overline{\text{SES}}_j = \sum_{i=1}^{n_j} (\text{SES})_{ij}$

Summary & Comparison: Random

	Null Model		Add cSES_{ij} & $(\overline{\text{SES}})_j$		Add all mirco level		All micro and macro	
	Value	SE	Value	SE	Value	SE	Value	SE
Random effects								
τ_0^2 (between)	8.55	1.07	2.64	.40	2.39	.37	1.29	.24
σ^2 (within)	39.15	.66	37.02	.62	35.89	.61	35.88	.61

Note:
$\text{cSES}_{ij} = (\text{SES})_{ij} - (\overline{\text{SES}})_j$
$\overline{\text{SES}}_j = \sum_{i=1}^{n_j} (\text{SES})_{ij}$

Summary: Random Intercept Models

Concepts covered:

- Within group dependency accounted for by random effect (i.e., R_{ij}).
- Between group differences accounted for by random intercept (i.e., U_{0j}).
- Central role of variance between and within groups.
- Effect of adding Level 1 and Level 2 predictors.
- Mean centering of level 1 predictors and adding the mean back in at Level 2.
- Intra-class correlation or ICC.
- HLMs with only random intercepts imply homoscedasticity.